The degree/diameter problem in planar graphs.

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The degree/diameter problem.

Question

What is the largest possible order $n_{\Delta,D}$ *of a graph with maximum degree* Δ *and diameter* D*?*

Directed version: with Δ^+.

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Lower bounds (when $D \geqslant 3$):

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Conjecture (Bollobás 1972)

For every $\varepsilon > 0$, $n_{\Delta,D} \geqslant (1-\varepsilon)\Delta^D$.

The degree/diameter problem on restricted classes.

- Vertex-transitive and Cayley graphs
- Bipartite graphs
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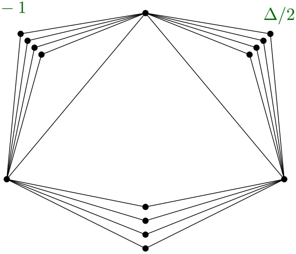
Theorem (Hell, Seyffarth 1993)

For every $\Delta \geqslant 8$,

$$\mathrm{np}_{\Delta,2} = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$$



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Theorem (Fellows, Hell, Seyffarth 1995)

For every $D \geqslant 1$*,*

$$\mathrm{np}_{\Delta,D} = \Theta\left(\Delta^{\lfloor \frac{D}{2} \rfloor}\right)$$

The degree/diameter problem on planar graphs.

Theorem (Tishchenko 2012)

For every D even and every Δ large enough, we have

$$\mathrm{np}_{\Delta,D} = \left\lfloor \frac{3\Delta}{2} \cdot \frac{(\Delta-1)^{\frac{D}{2}}-1}{\Delta-2} \right\rfloor + 1.$$

The degree/diameter problem on diameter 3 planar graphs.

Theorem (Fellows, Hell, Seyffarth 1995)

For every $\Delta \geqslant 8$,

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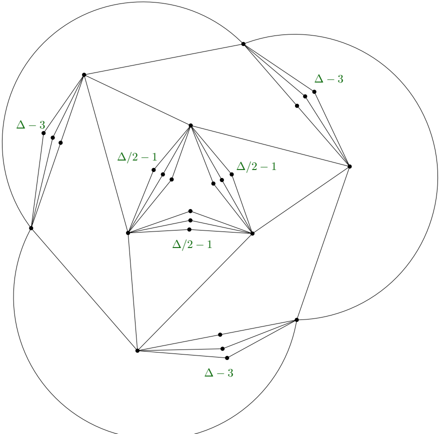

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- G is a quadrangulation of the plane (Dalfo, Huemer, Salas 2016);
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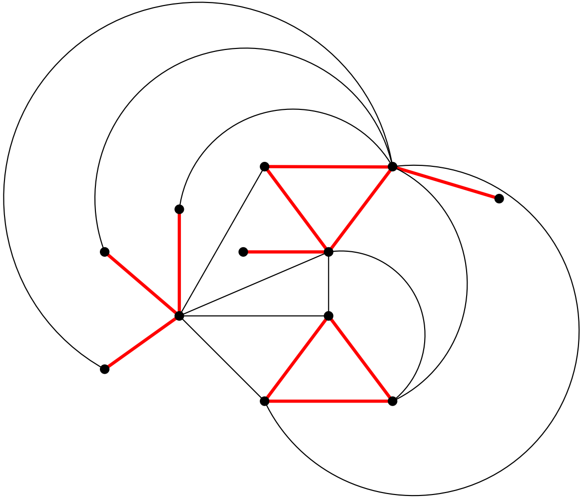
Theorem (Dailly, Darmon, G., Hilaire, Valicov 2025+)

$$\mathrm{np}_{\Delta,3} = \frac{9}{2}\Delta + O(1)$$

+ Structural "characterization" of extremal classes.

Constructions of large planar graphs with $D = 3$.

A neighbouring set of edges in a graph G is a set $F \subseteq E(G)$ s.t. every two edges $e, e' \in F$ are at distance at most 1 in G.



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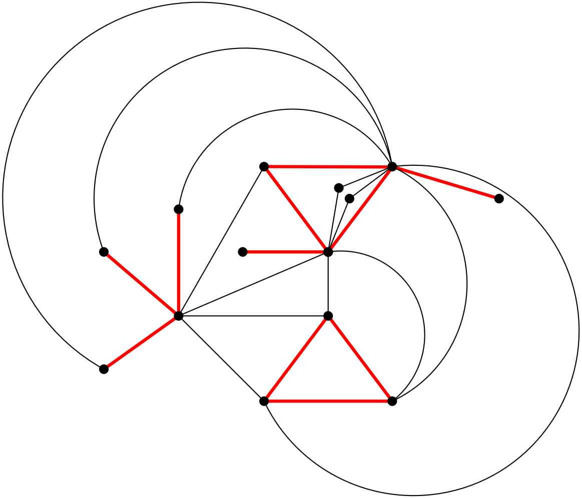

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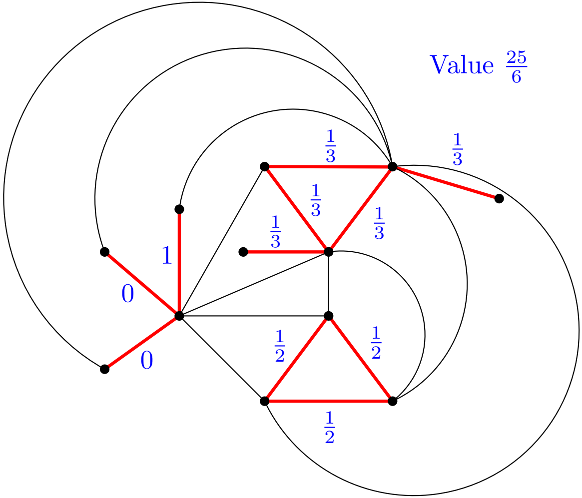

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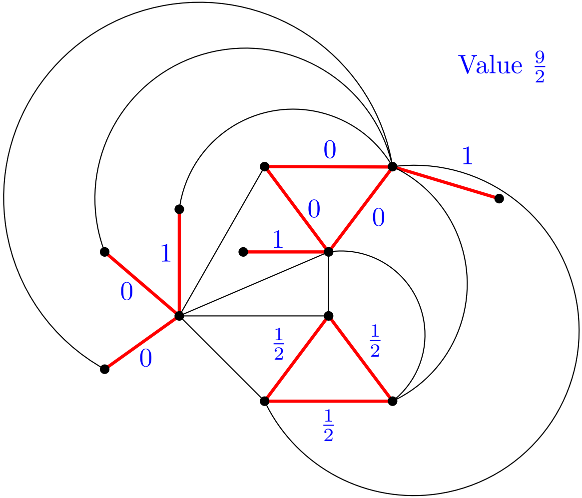

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For every K_5-minor free graph G and every neighbouring set of edges $F \subseteq E(G)$ we have

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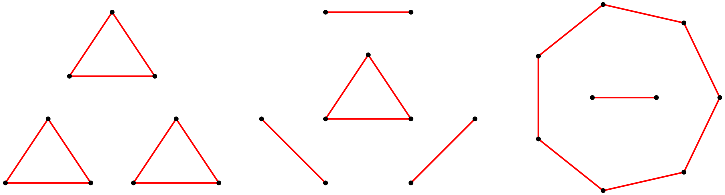

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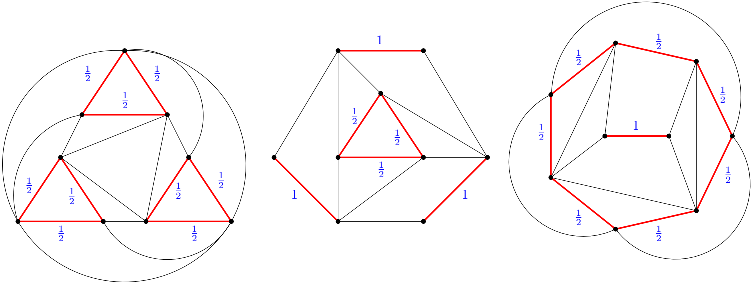

Fractional matchings/vertex covers

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$\rho : V(G) \to \mathbb{R}_{\geqslant 0}$ is a fractional vertex cover of G if for every $uv \in E(G)$, $\mu(u) + \mu(v) \geqslant 1$.
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Theorem (Strong duality of LP)

For every graph we have $\mu(G) \leqslant \mu_f(G) = \rho_f(G) \leqslant \rho(G)$.

Proof sketch that $\mu_f \leqslant \frac{9}{2}$

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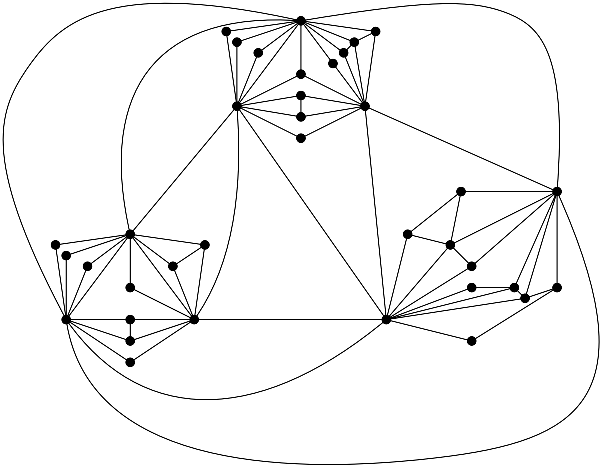
Blackboard.

From degree/diameter to neighbouring sets.



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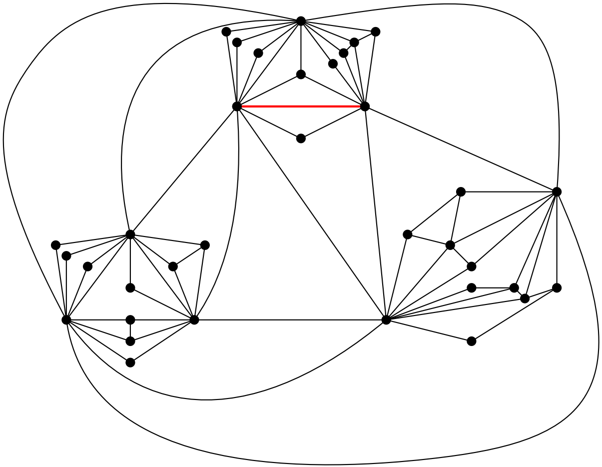

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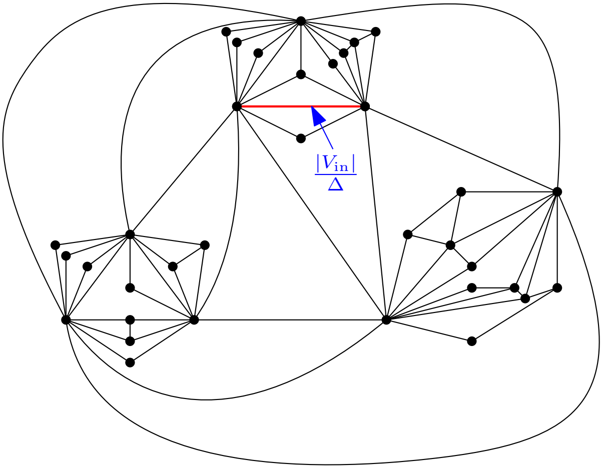

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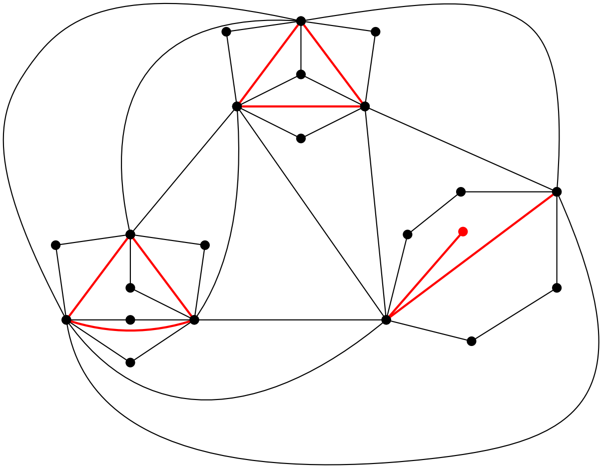

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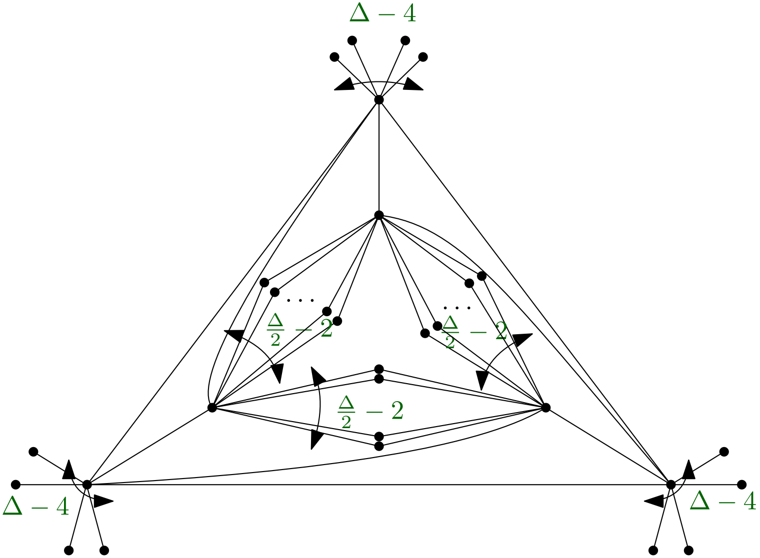

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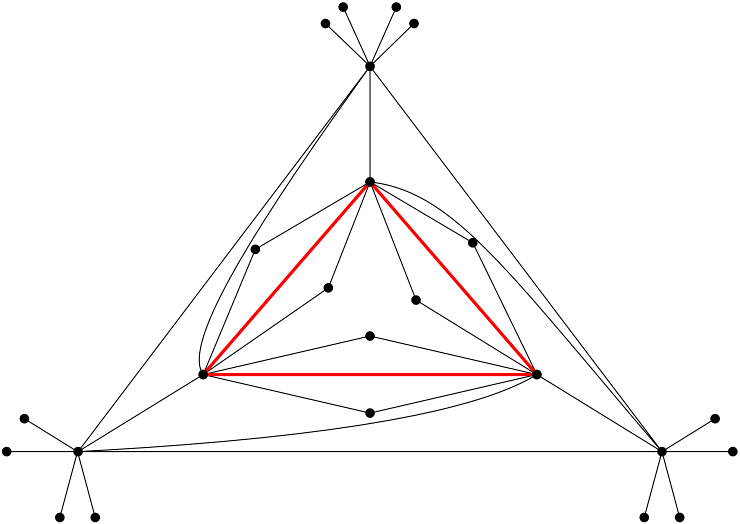

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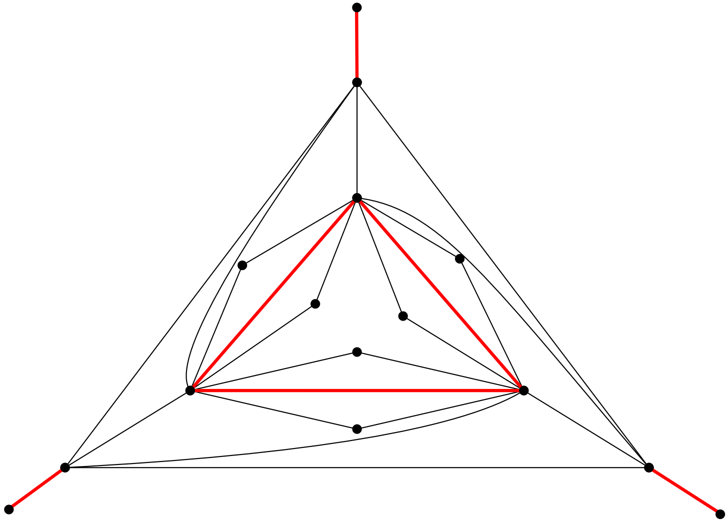
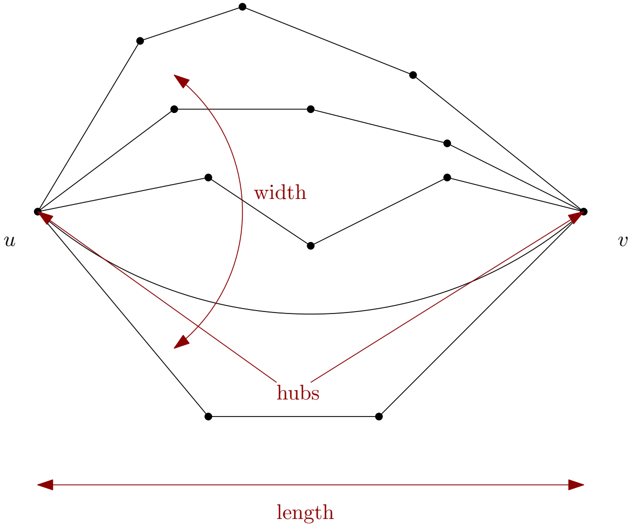


Lanterns

A lantern L with hubs u, v is a collection of internally disjoint (u, v)-paths.



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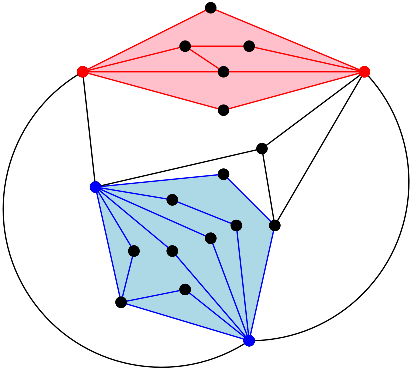

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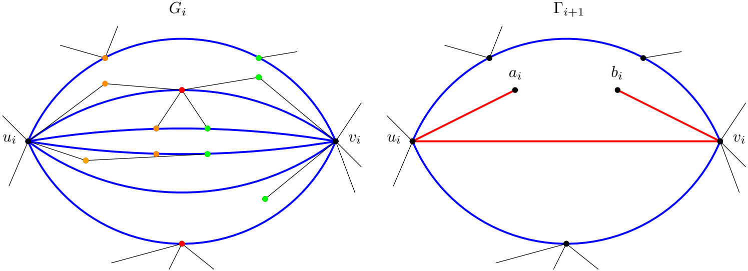
Lemma (Lantern extraction)

Let G be a planar graph of diameter 3. If G has a lantern L of width 78, then L has a dominating sublantern of length at most 3, which can be "safely" emptied.

More details on blackboard.

General strategy

Start from graph $G_0 := G$; auxiliary graph Γ_0. While G_i contains a lantern of width 78, take L' given by the extraction lemma and empty it. Gives a graph G_{i+1}. Empty also L' in Γ_i and add properly the associated red edges to obtain Γ_{i+1}.



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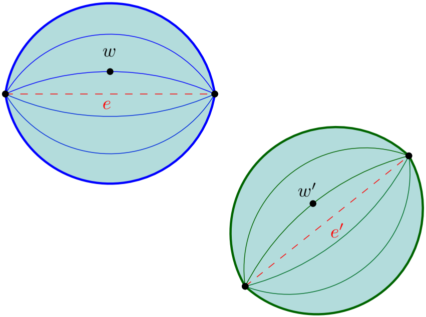
At each step i, the set F_i of red edges is a neighbouring set of edges of Γ.



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Proposition

At the end of the procedure, we are left with a planar graph G_i of diameter 3 without lantern of width 78, and with F_i neighbouring set of edges of Γ_i such that

$$|V(G) \setminus V(G_i)| \leqslant \mu_f(\Gamma_i[F]) \cdot \Delta(G).$$

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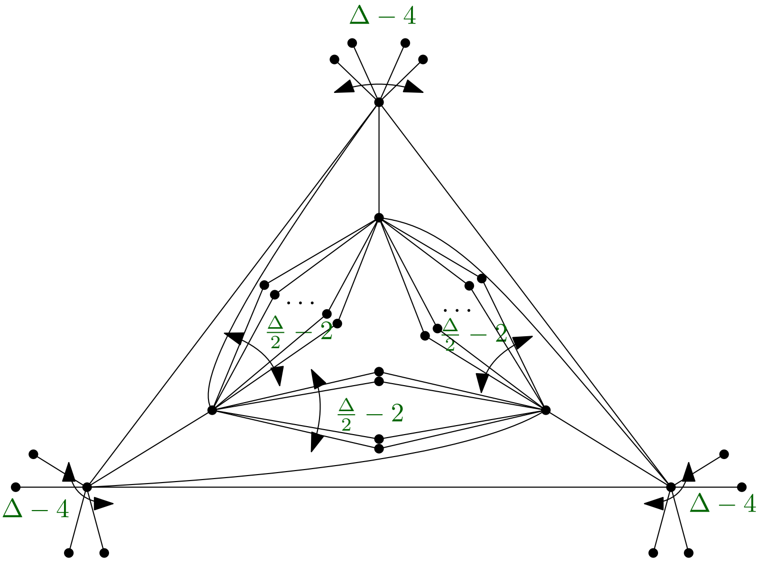

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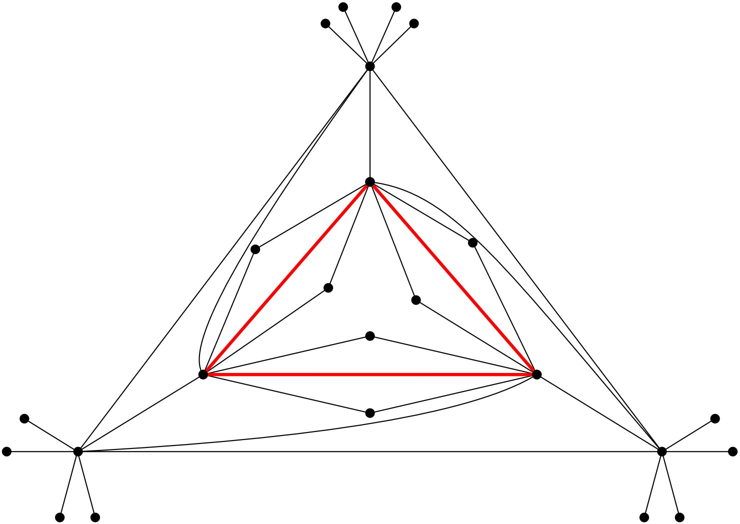

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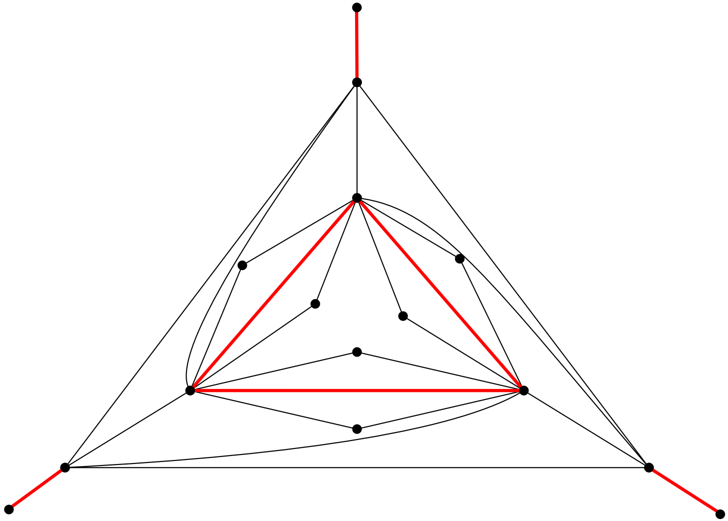

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Every planar graph G of diameter 3 has a dominating set of size 10.

D $|D| \leq 10$

$G_0 - D$

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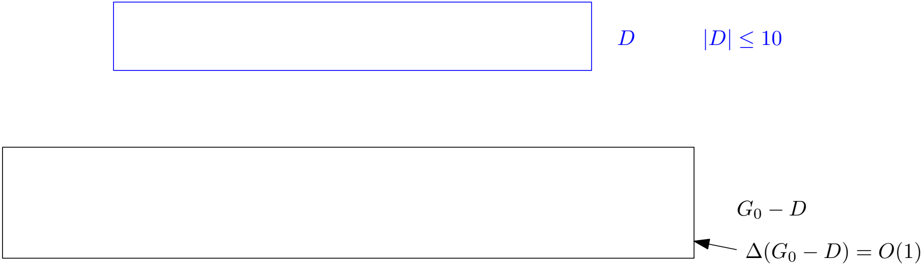

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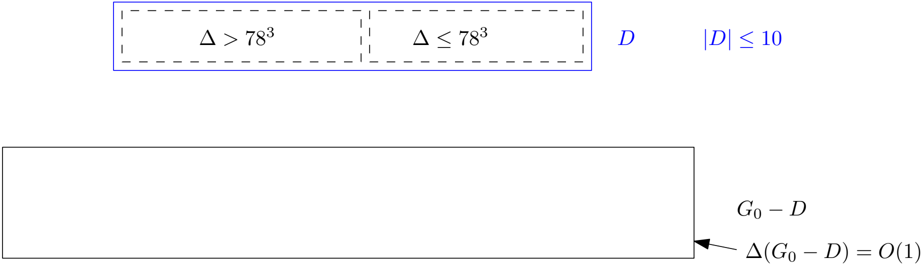

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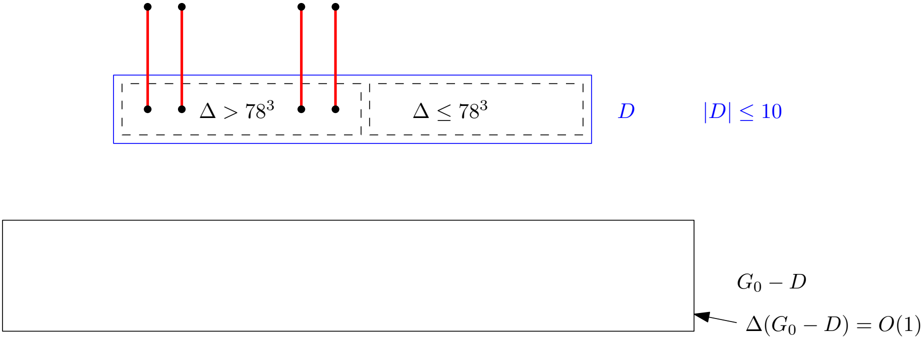

Conclusion and further directions

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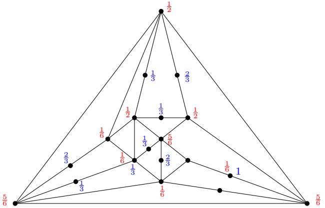
- More general ways to reduce to fractional matching problems in other classes, e.g. bounded genus graphs?
- In general, $|V(G)| \leqslant \gamma_f(G)(\Delta + 1)$. If G is planar, large enough with diameter 3, then $\gamma(G) \leqslant 6$ (Dorfling, Goddard, Henning 2006). What is the maximal possible value for $\gamma_f(G)$?



Thank you for your attention.