Coarse cops and robber games.

Ugo Giocanti,
joint work with Louis Esperet and Harmender Gahlawat

Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

Jagiellonian TCS seminar

Cops and robber games in finite graphs.

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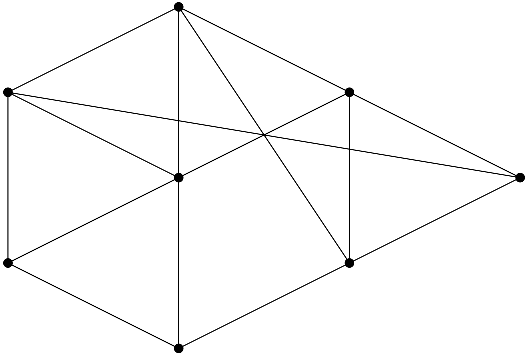
Full information game on a fixed finite graph G.



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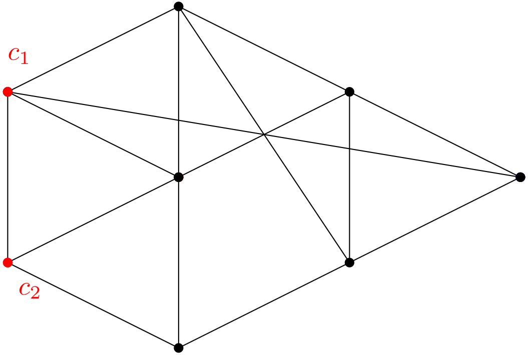

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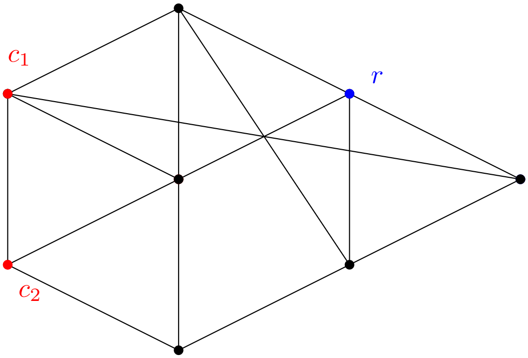

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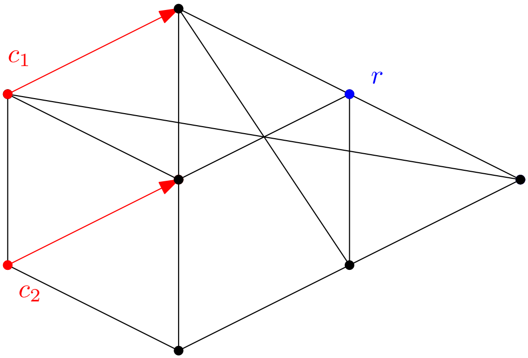

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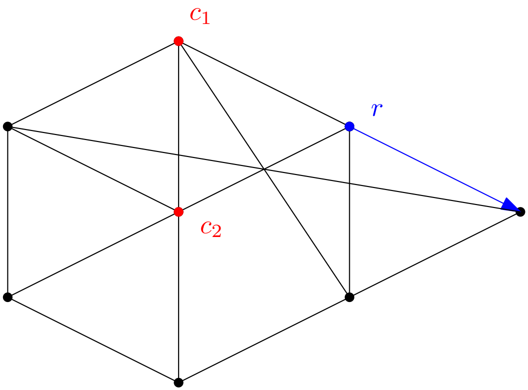

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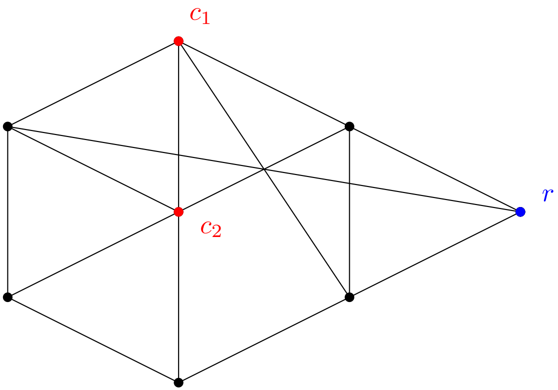

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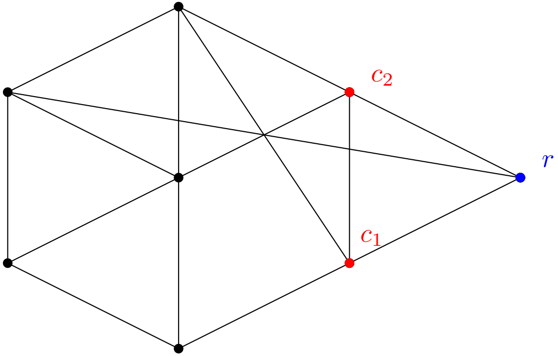

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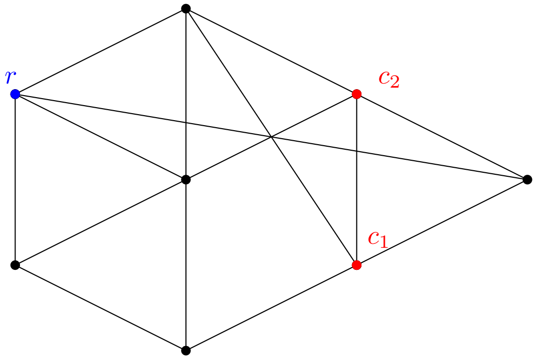

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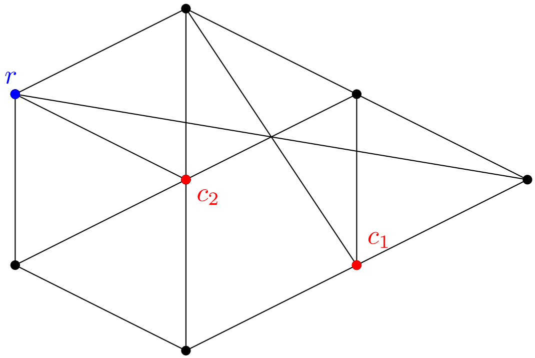

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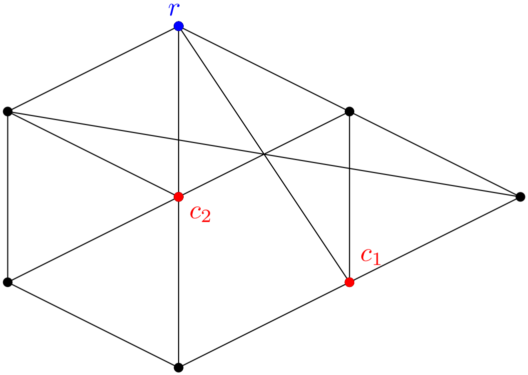

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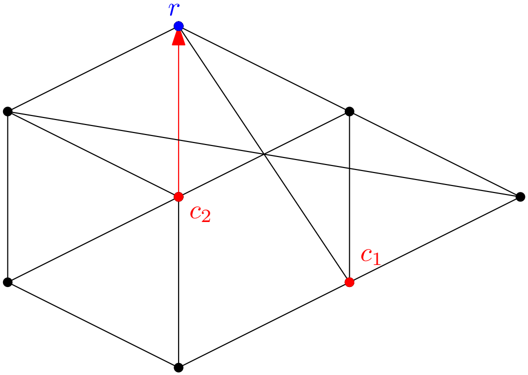

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A collection of cops $\{c_1, \ldots, c_m\}$ guards a subgraph H of G if these cops have a strategy in which, from some step, they can occupy vertices of H, and ensure that the robber will not be able to occupy a vertex of H anymore.

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Corollary (Aigner, Fromme 1984)

For every planar graph G, $\mathrm{cop}(G) \leqslant 3$.

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- Pursuit-evasion variant in geodesic spaces (Mohar 2021).
- Coarse variants introduced by Lee, Martínez-Pedroza, and Rodríguez-Quinche (2023).

Coarse cops and robber games.

G: fixed infinite graph.

Settings: k cops $c_1, \dots, c_k$ with speed $s_c \geqslant 1$ and reach $\rho \geqslant 1$. One robber with speed $s_r \geqslant 1$. $B = B(v_0, R)$: fixed ball of finite radius $R \geqslant 1$.

Step 0: $c_1, \dots, c_k$ choose positions $v_1, \dots, v_k \in V(G)$. Then the robber r chooses a position $v \in V(G)$ at distance $> \rho$ from the cops.

Step $i \geqslant 1$: every cop can move using any path of length at most s_c. If after their moves, r is at distance at most ρ from a cop, the cops immediately win. Otherwise, the robber can move using any path of lenth at most s_r.

Goal of robber: enter in B infinitely many times without being captured.

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G is CopWin(k, s_c, ρ, s_r, R) if k cops have a winning strategy with parameters (k, s_c, ρ, s_r, R) for every ball B of radius R.

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Weak game:

1. $c_1, \ldots, c_k$ choose s_c and ρ.
2. r chooses s_r and R.

Strong game:

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Weak (resp. Strong) cop number wCop$(G) \in \mathbb{N} \cup \{\infty\}$ (resp. sCop$(G) \in \mathbb{N} \cup \{\infty\}$) of G: infimum over the $k \geqslant 1$ such that k cops win the weak (resp. strong) game.

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- If G is connected, then the center v_0 of B does not matter.
- $\mathrm{sCop}(G) \leqslant \mathrm{wCop}(G)$.

Quasi-isometries

Two (infinite) graphs G, H are quasi-isometric if there exist $f : V(H) \to V(G)$ and constants $A, B, C > 0$ such that:

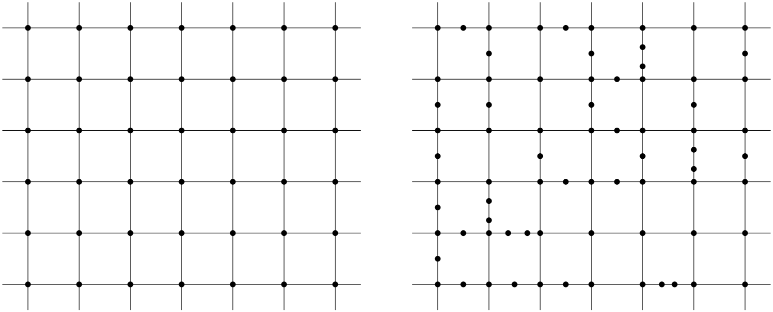
(1)

$$\forall x, y \in V(H), \frac{1}{A} d_H(x, y) - B \leqslant d_G(f(x), f(y)) \leqslant A d_H(x, y) + B,$$

(2) for every $y \in V(G)$, there exists $x \in V(H)$ such that $d_G(y, f(x)) \leqslant C$.

If only (1) holds, then f is a quasi-isometric embedding.

Quasi-isometries



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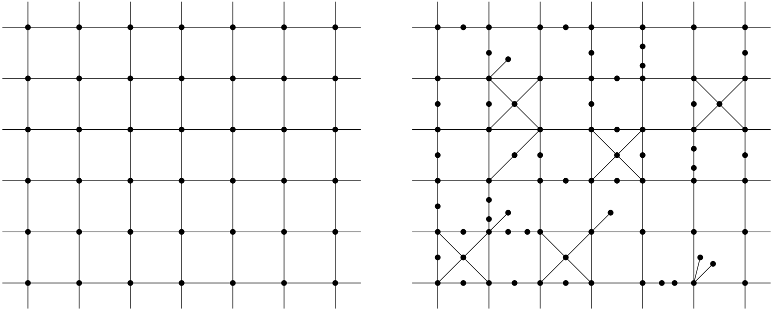

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Quasi-isometry invariance

Theorem (Lee, Martínez-Pedroza, Rodríguez-Quinche 2023)

If G *and* H *are quasi-isometric, then* $\mathrm{wCop}(G) = \mathrm{wCop}(H)$ *and* $\mathrm{sCop}(G) = \mathrm{sCop}(H)$.

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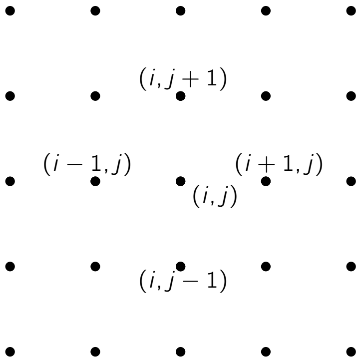
Theorem (Esperet, Gahlawat, G.)

If $f : H \to G$ is a quasi-isometric embedding, then $\mathrm{wCop}(H) \leqslant \mathrm{wCop}(G)$ *and* $\mathrm{sCop}(H) \leqslant \mathrm{sCop}(G)$.

Proof idea: blackboard.

$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



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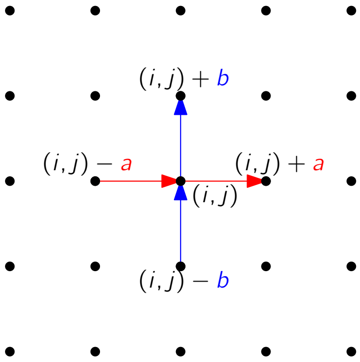

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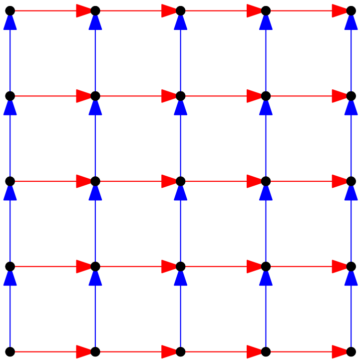

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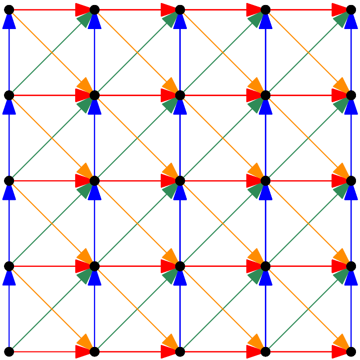

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$c := (1, 1)$, $d := (1, -1)$

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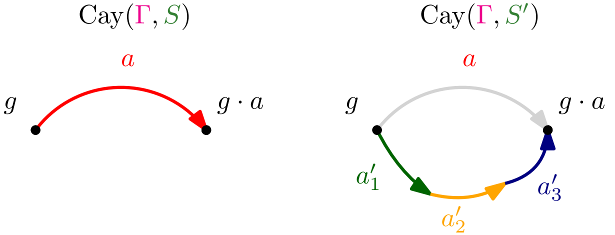
→ How are $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, S')$ related for two different finite generating sets S, S'?



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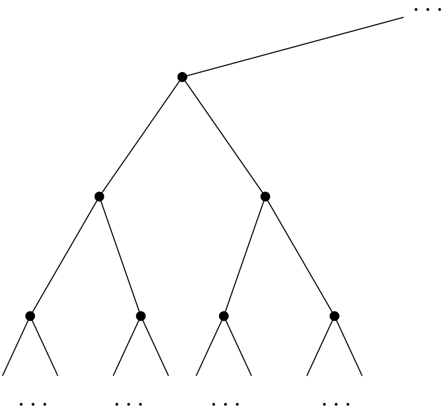
$\rightarrow$ How are $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, S')$ related for two different finite generating sets S, S'?

Proposition

Two Cayley graphs of a same group Γ are quasi-isometric.

$\rightarrow$ One can define the weak/strong cop number of finitely generated groups.

Examples.



Lemma

For every tree T, $\mathrm{wCop}(T) = \mathrm{sCop}(T) = 1$*.*

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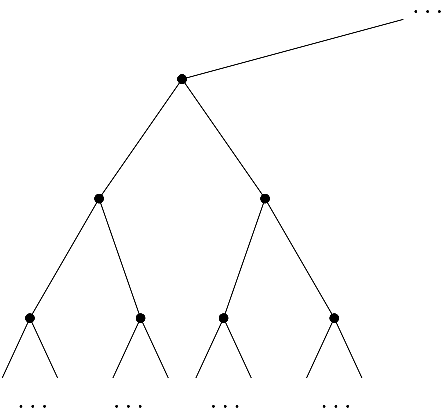

Lemma

For every tree T, $\mathrm{wCop}(T) = \mathrm{sCop}(T) = 1$*. In particular, if G is quasi-isometric to a tree, then* $\mathrm{wCop}(G) = \mathrm{sCop}(G) = 1$*.*

Examples.

Proposition

If a locally finite connected graph G is quasi-isometric to a graph of treewidth at most t, then $\mathrm{wCop}(G) \leqslant t + 1$.

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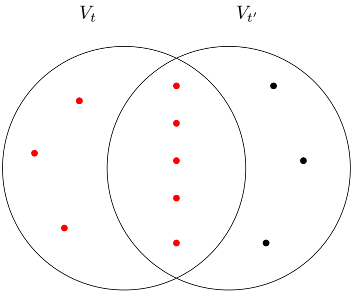

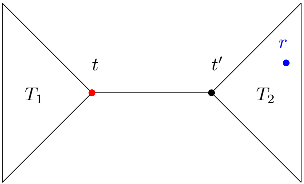

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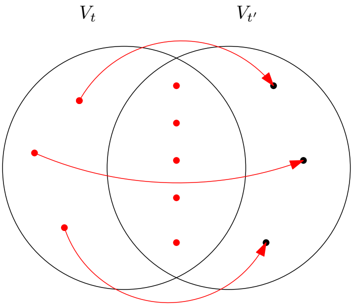

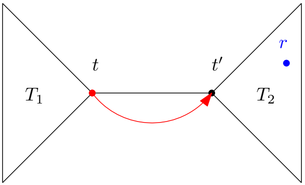

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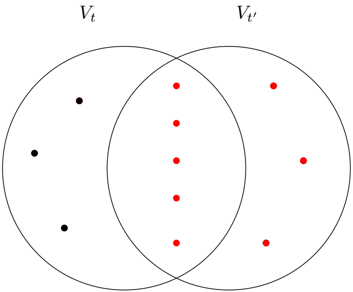

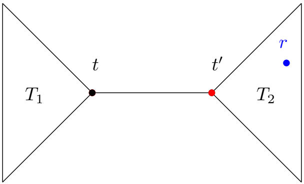

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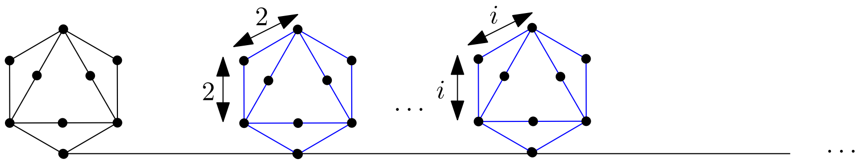
The bound is tight.



Here: $\mathrm{tw}(G) = 2$, while $\mathrm{wCop}(G) \geqslant 3$.

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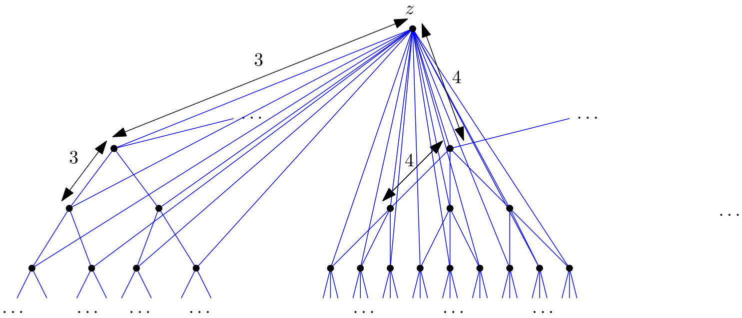

A graph G with $\mathrm{tw}(G) = 2$ and sCop(G) = wCop$(G) = \infty$.

H: finite graph. A D-fat model of H in G is a family $((V_x)_{x\in V(H)}, (P_{xy})_{xy\in E(H)})$ where:

- for each $x \in V(H)$, V_x induces a connected subgraph of G,
- P_{xy} is a path with one endpoint in V_x and the other in V_y,
- for every $x \neq y \in V(H)$, $d_G(V_x, V_y) \geqslant D$,
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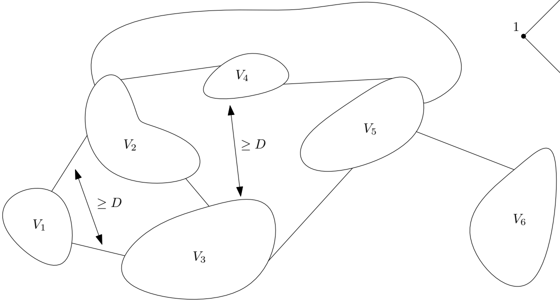

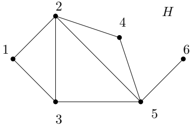

wCop and asymptotic minors.

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H is an asymptotic minor of G, denoted $H \preceq_\infty G$, if there is a D-fat model of H in G for every $D \geqslant 1$.

Remark (Georgakopoulos, Papasoglou 2023)

For every fixed graph H, and every two quasi-isometric graphs G, G',

$$(H \preceq_\infty G) \Leftrightarrow (H \preceq_\infty G').$$

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Theorem (Esperet, Gahlawat, G. 2025)

For every finite graph H, if $H \preceq_\infty G$, then

$$\mathrm{tw}(H) \leqslant \mathrm{wCop}(G).$$

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For every finite graph H, if $H \preceq_\infty G$, then

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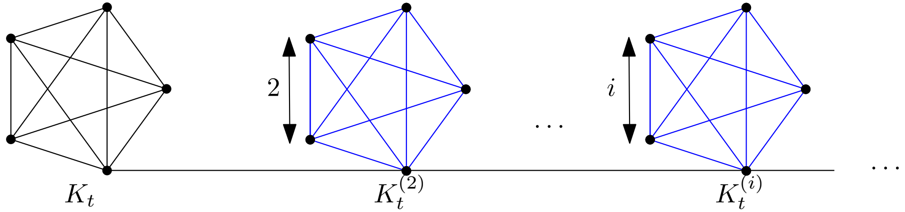
The bound is tight.



→ G is such that $K_t \preceq_\infty G$ and $\mathsf{wCop}(G) = \mathrm{tw}(K_t) = t - 1$.

wCop and asymptotic minors.

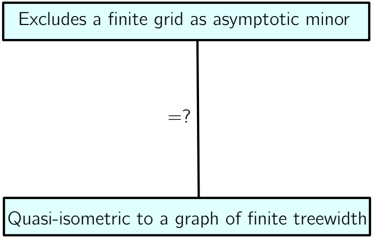
Theorem (Esperet, Gahlawat, G.)

For every finite graph H, if $H \preceq_\infty G$, then

$$\mathrm{tw}(H) \leqslant \mathsf{wCop}(G).$$

Conjecture (Georgakopoulos, Papasoglou 2023)

A graph G is quasi-isometric to a graph of finite treewidth if and only if it excludes a finite $k \times k$ grid as an asymptotic minor.



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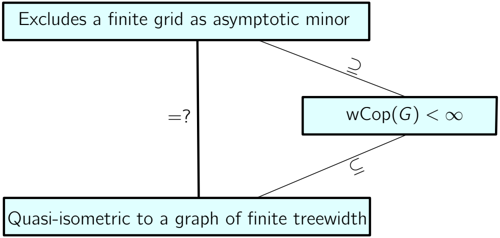

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$$\sup\{\mathrm{tw}(H) : H \preceq_\infty G\} \leqslant \mathsf{wCop}(G) \leqslant \inf\{\mathrm{tw}(G')+1 : G' \text{ quasi-isometric to } G\}.$$

Question

Does there exists $f, g : \mathbb{N} \to \mathbb{N}$ such that:

- $\mathsf{wCop}(G) \leqslant f(\sup\{\mathrm{tw}(H) : H \preceq_\infty G\})$?
- $\inf\{\mathrm{tw}(G') + 1 : G' \text{ quasi-isometric to } G\} \leqslant g(\mathsf{wCop}(G))$?

If yes, can we choose $f, g = \mathrm{id}_{\mathbb{N}}$?

Proof that wCop(G) $> \frac{\mathrm{tw}(H)+1}{2}$.

A haven of order k in a graph G is a mapping β mapping each $X \in \binom{V(G)}{\leqslant k}$ to some component $\beta(X)$ of $G - X$, such that for every $X, Y \in \binom{V(G)}{\leqslant k}$, $\beta(X)$ and $\beta(Y)$ touch. $\mathrm{bn}(G)$ denotes the largest order of a haven in G.

Theorem (Seymour, Thomas 1993)

For every finite graph G, we have $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$.

Proof that $\mathsf{wCop}(G) > \frac{\mathrm{tw}(H)+1}{2}$.

Let $H \preceq_\infty G$, $k := \lfloor \frac{\mathrm{tw}(H)+1}{2} \rfloor$ and β be a haven of order $\mathrm{tw}(H) + 1 \geqslant 2k$.
Winning strategy for the robber when playing against k cops.

Let s_c, ρ denote the speed and reach of the k cops. Consider a D-fat model $((V_x)_{x \in V(H)}, (P_e)_{e \in E(H)})$ of H in G, with $D := 2(s_c + \rho + 1)$.

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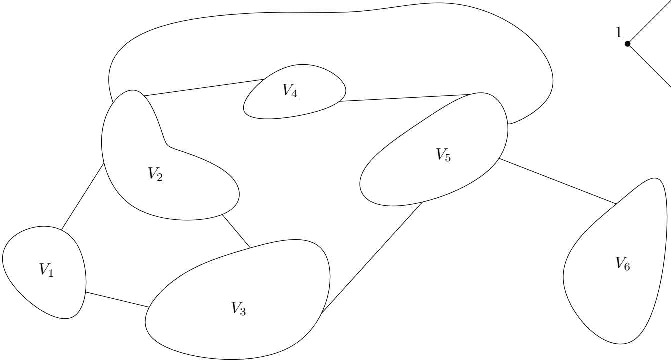

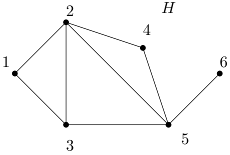

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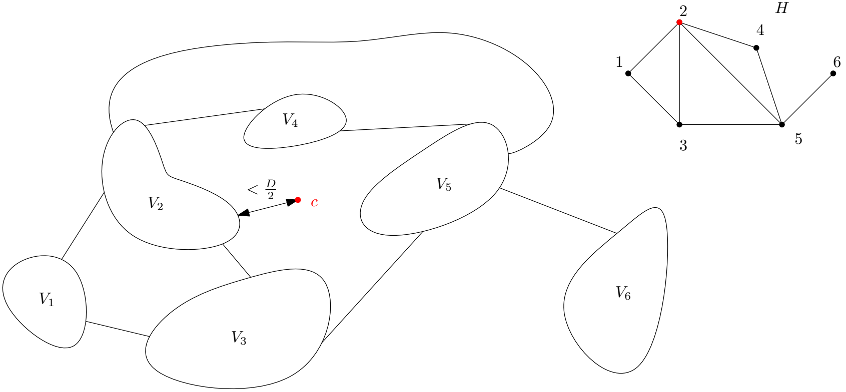

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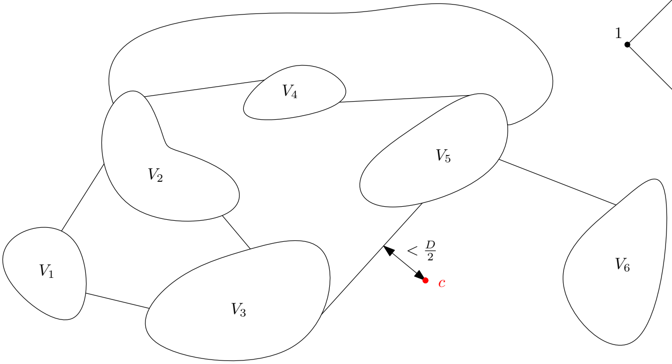

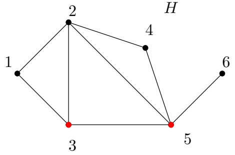

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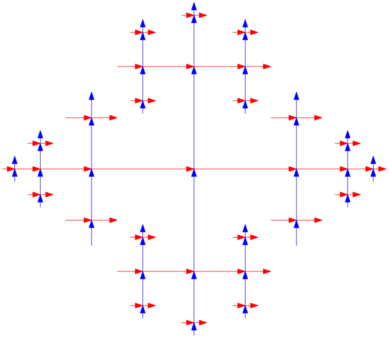

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Question (Lee, Martínez-Pedroza, Rodríguez-Quinche 2023)

Do we have for every finitely generated group Γ, that $\mathsf{wCop}(\Gamma) \in \{1, \infty\}$?

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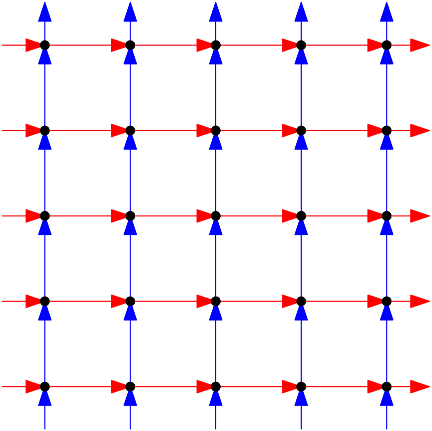
For every finitely generated group Γ, we have $\mathsf{wCop}(\Gamma) \in \{1, \infty\}$.

sCop and hyperbolicity.

Theorem (Esperet, Gahlawat, G. and Appenzeller, Klinge 2025)

We have $\mathrm{sCop}(\mathbb{Z}^2) = \infty$.



G is δ-hyperbolic ($\delta \geqslant 0$) if for every $x, y, z \in V(G)$, and every shortest paths P_1, P_2, P_3 connecting respectively x to y, y to z and z to x, P_i is at distance at most δ from $\bigcup_{j \neq i} P_j$ in G.
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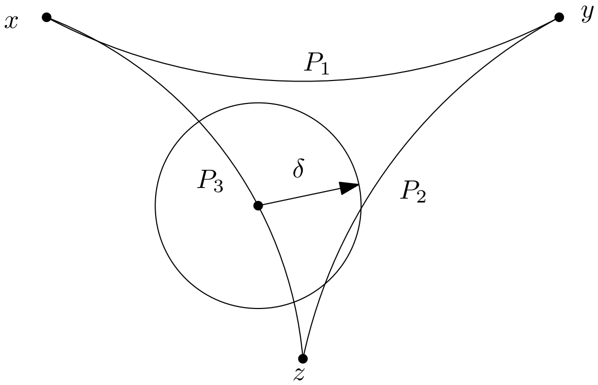

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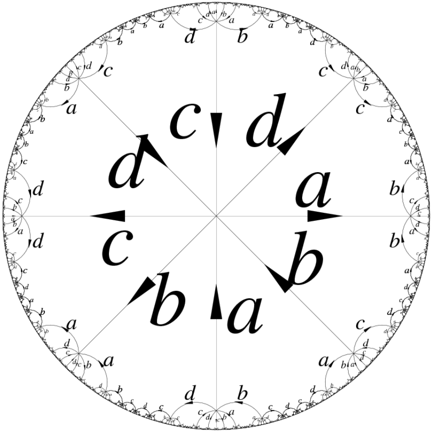

Image source: Yann Ollivier. A primer to geometric group theory.
http://www.yann-ollivier.org/maths/primer.php

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Every hyperbolic graph G satisfies $\mathrm{sCop}(G) = 1$.

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Theorem (Chalopin, Chepoi, Nisse, Vaxès 2011, Chalopin, Chepoi, Papasoglou, Pecatte 2014)

Let G be a finite graph.

- *If G δ-hyperbolic, then it is (s_c, s_r)-cop-win, for every s_c, s_r such that $s_r - s_c \leqslant 2\delta$.*
- *If G is (s_c, s_r)-cop-win for some s_c, s_r with $s_c < s_r$, then it is δ-hyperbolic for some $\delta = O({s_r}^2)$.*

sCop and hyperbolicity.

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Proof of $\Leftarrow$.

Theorem ("Linear isoperimetric inequality". Gromov 1987, Bowditch 1991)

If G is hyperbolic, then there exists $D, K \geqslant 1$ such that for every cycle C such that $|C| \geqslant K$, no subpath of C of length at least D is a shortest path in G.

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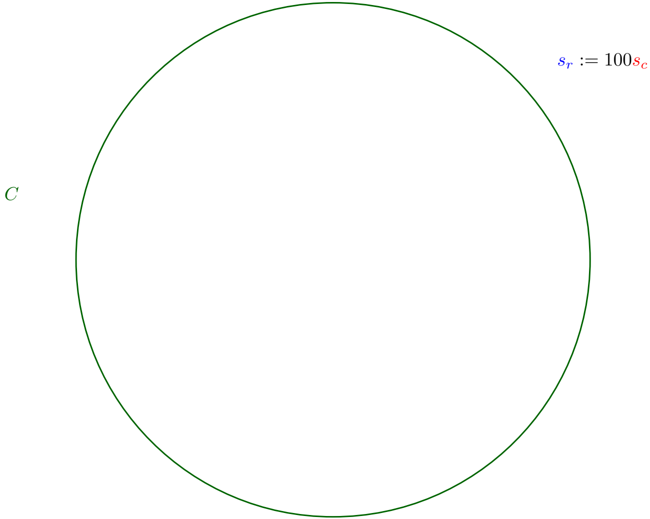
$\rightarrow$ Assume that G is not hyperbolic. Let s_c be the speed of the cop, and let $s_r := 100 s_c$. Let ρ be the reach of the cop. The robber then chooses a cycle C of length $\geqslant 400\rho s_c$ such that all subpaths of length $200\rho s_c$ are geodesics.

sCop and hyperbolicity.



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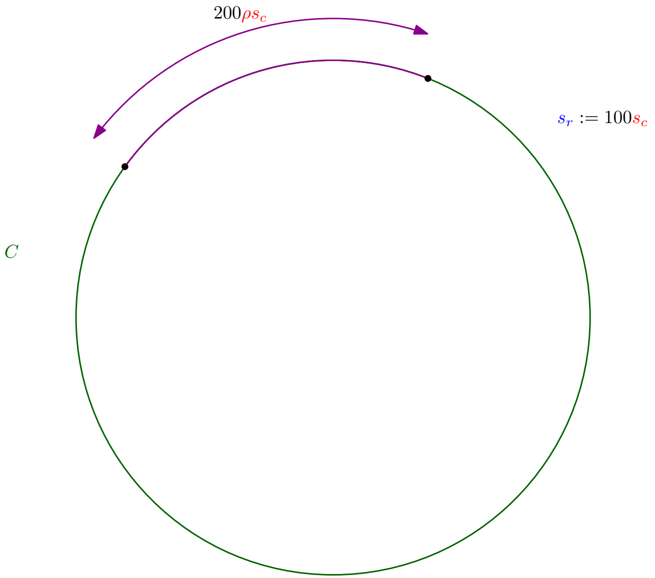

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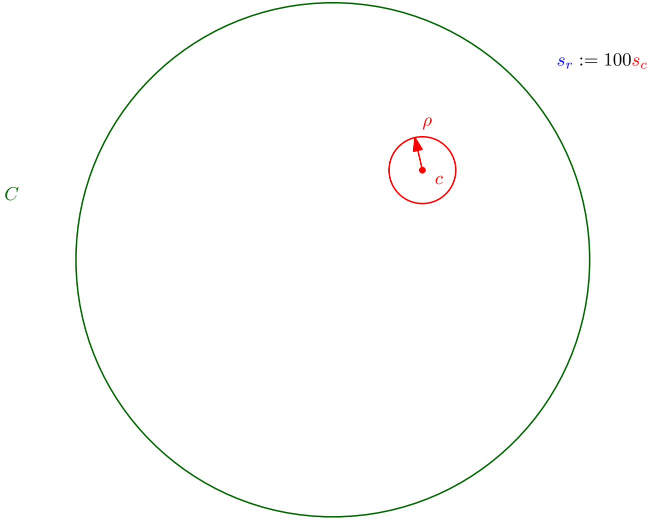

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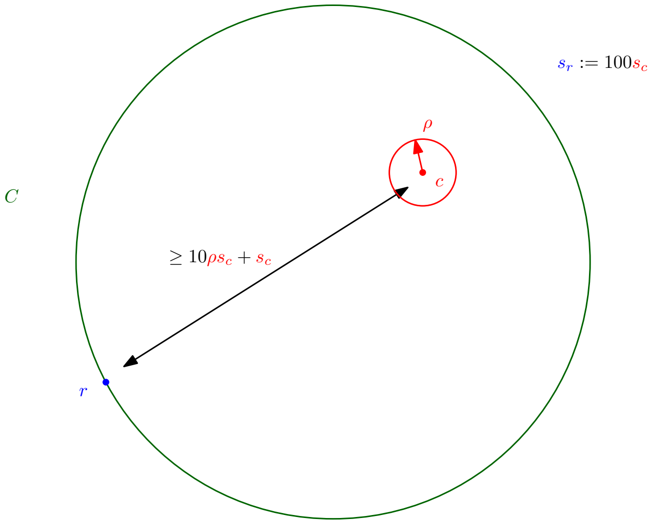

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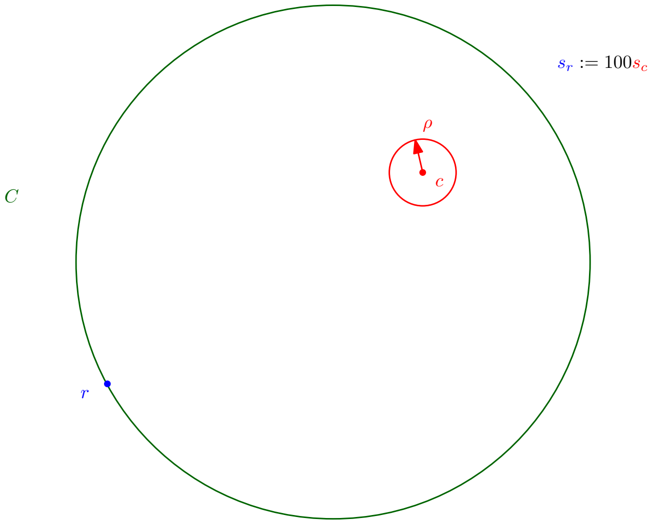

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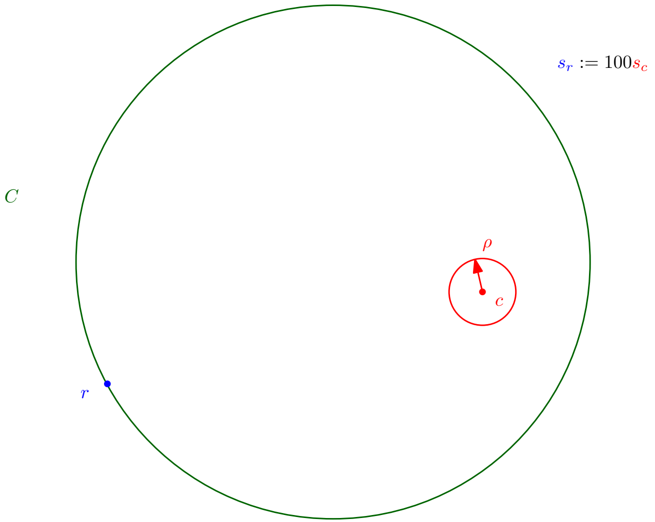

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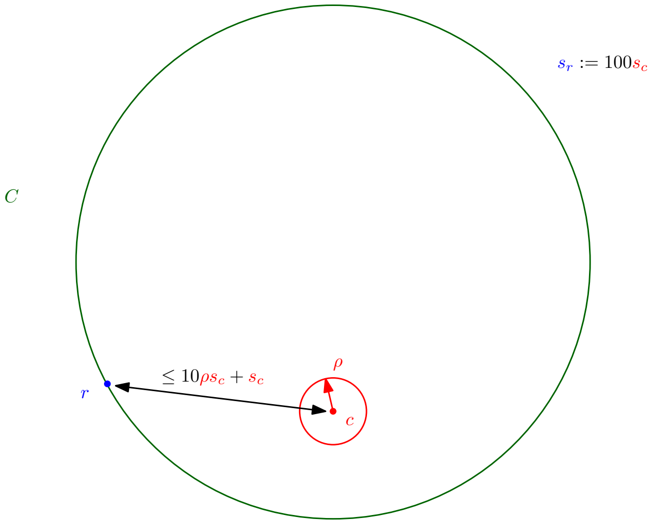

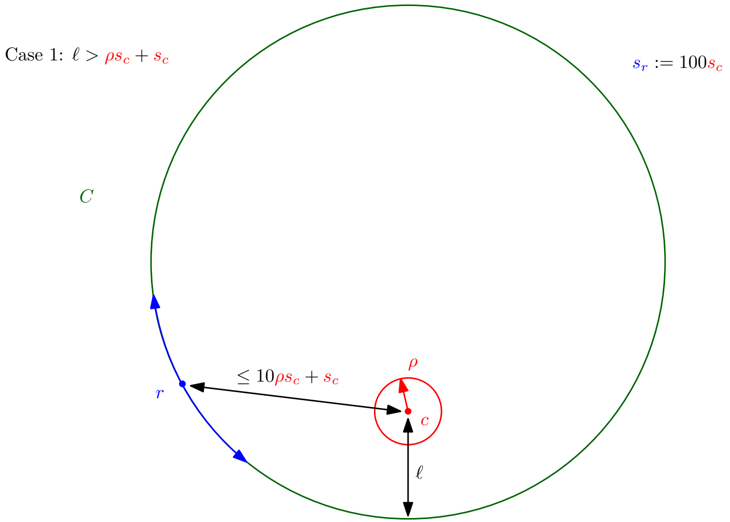
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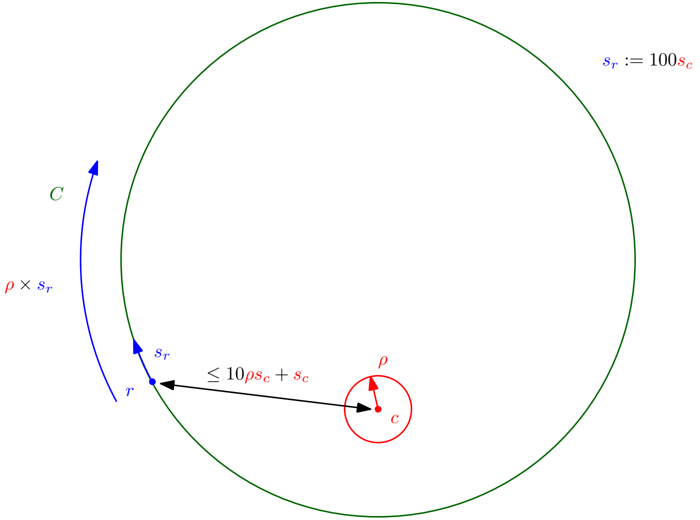

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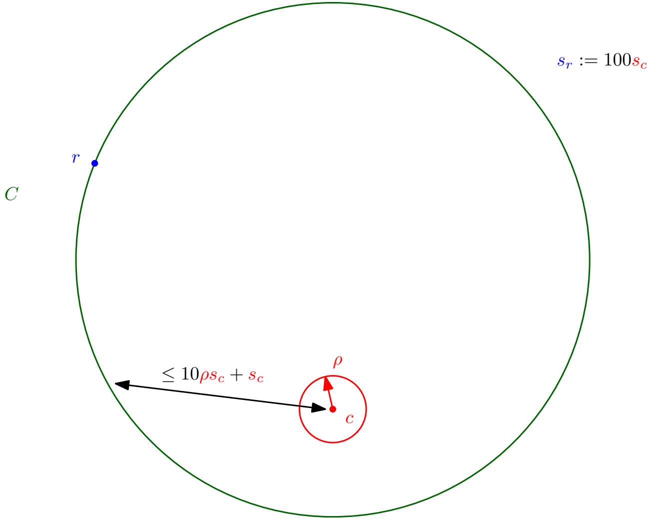

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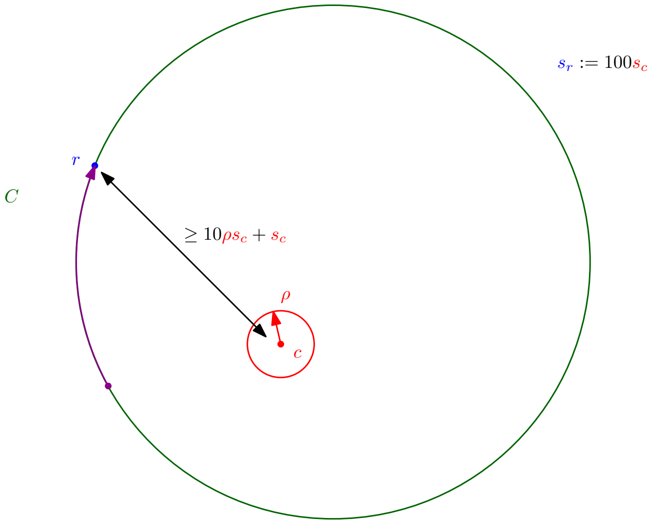

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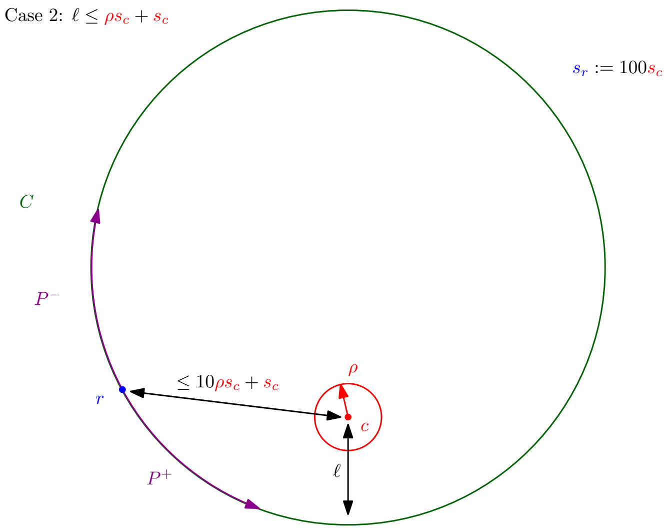

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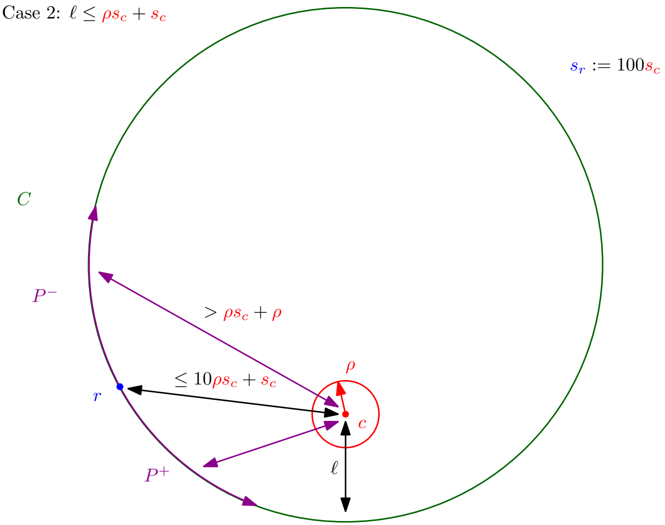

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Conclusion

For graphs:

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- Graphs with wCop equal to 1 are exactly graphs quasi-isometric to trees.
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- A finitely generated group has wCop equal to 1 if and only if it has finite wCop, if and only if it is virtually free.
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Dziękuję bardzo.