Structure of graphs edge-coverable by a few isometric paths or trees.

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Joint work with Julien Baste, Lucas De Meyer, Etienne Objois and Timothé Picavet

Jagiellonian University

JGA 2025, Paris.

Covering problems

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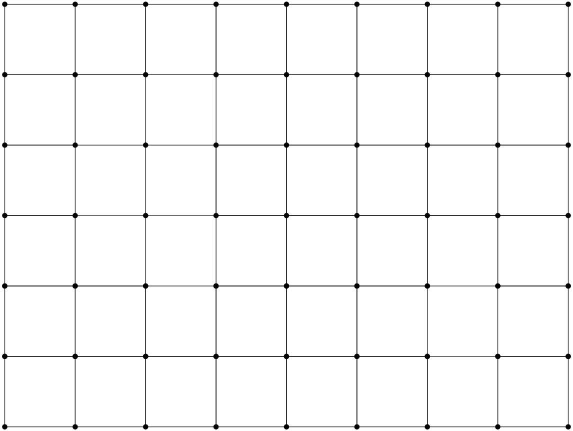

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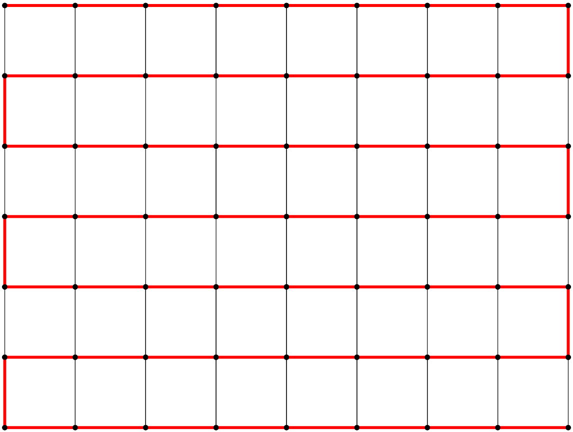

A vertex-covering of the grid by one path.

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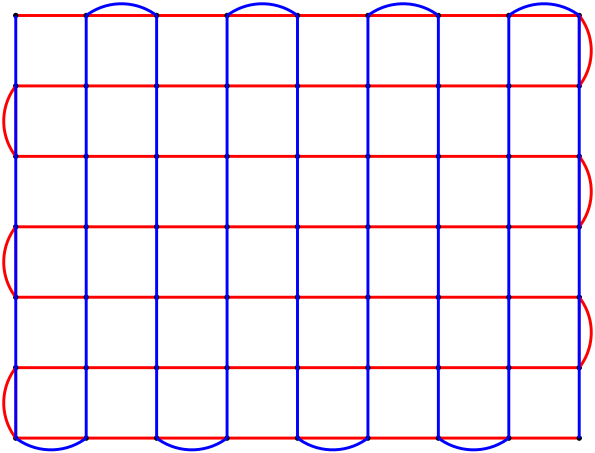

An edge-covering of the grid by two paths.

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Question

Let G be a graph vertex/edge-coverable by a few graphs from $\mathcal{H}$. What is the structure of G? How close is it from a graph of $\mathcal{H}$?

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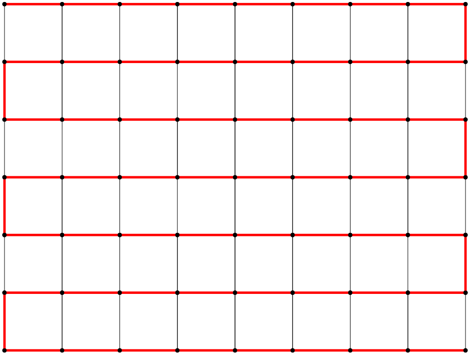
H is an isometric subgraph of G if all shortest paths in H are also shortest paths in G.



A vertex-covering of the grid by one path which is NOT isometric.

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Let G be a graph vertex/edge-coverable by a few isometric subgraphs from $\mathcal{H}$. What is the structure of G? How close is it from a graph of $\mathcal{H}$?

The good news: coverings by isometric paths

Graphs coverable by a few isometric subpaths look like a path!

Theorem (Dumas, Foucaud, Perez, Todinca 2024)

Let G be a graph vertex-coverable by k shortest paths. Then $\mathrm{pw}(G) = O(k \cdot 3^k)$. *Moreover, if G is edge-coverable by k shortest paths, then* $\mathrm{pw}(G) = O(3^k)$.

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Let G be a graph edge-coverable by 3 shortest paths. Then $\mathrm{pw}(G) \leqslant 3$.

Original proof of $O(3^k)$

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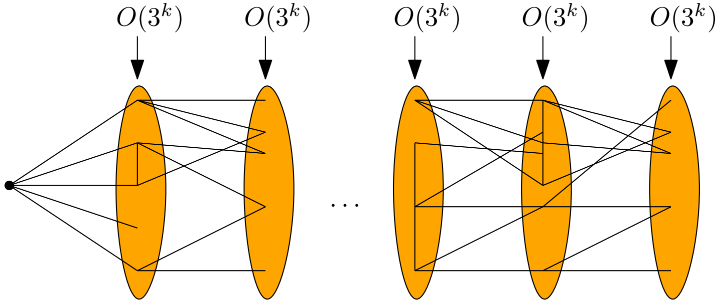

BFS layering

Some proof ideas

Fix G edge-covered by shortest paths $P_1, \dots, P_k$.
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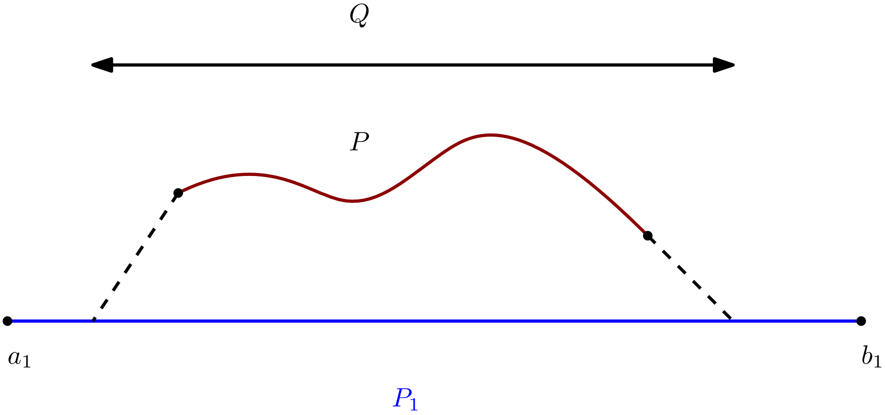

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If G is coverable by ℓ paths all parallel to P_1, then $\mathrm{pw}(G) \leqslant \ell$.

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Theorem (Main)

There exists a set $X \subseteq V(G)$ of size $O(k^3)$ such that every component of $G - X$ intersecting P_1 consists of $O(k)$ paths all parallel to P_1

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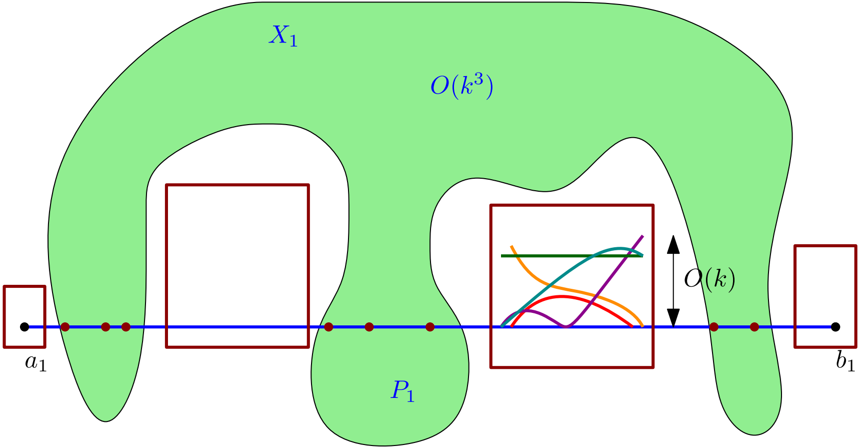

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For each $i \in \{1, \ldots, k\}$*, there exists a set* $X_i \subseteq V(G)$ *of size* $O(k^3)$ *such that every component of* $G - X_i$ *intersecting* P_i *consists of* $O(k)$ *paths all parallel to* P_i

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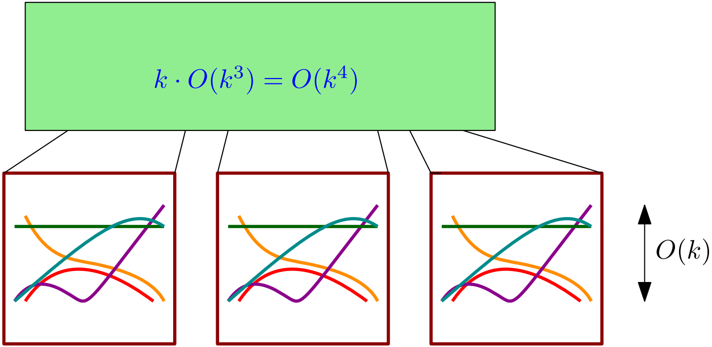

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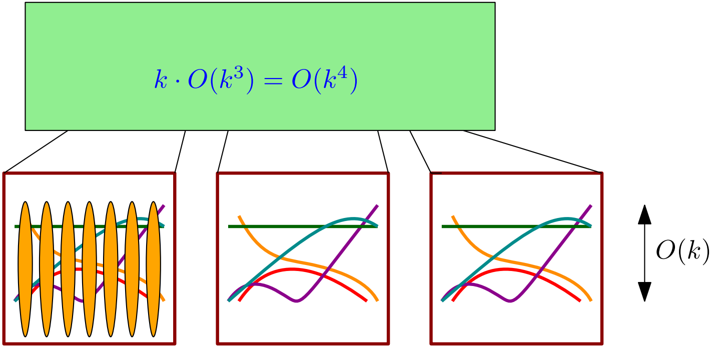

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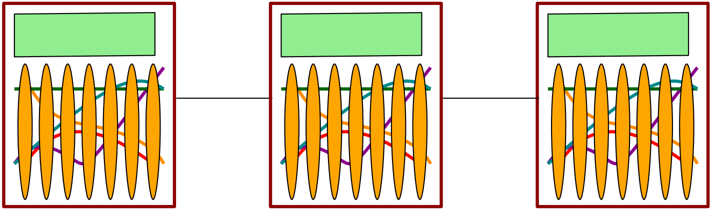

More general isometric covering problems? Trees?

Structure of graphs coverable by a few isometric trees?

Theorem (Ball, Bell, Guzman, Hanson-Colvin, Schonsheck 2017)

Every graph vertex-coverable by k isometric subtrees has cop number at most k.

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Does there exists $f : \mathbb{N} \to \mathbb{N}$ such that every graph coverable by k isometric subtrees has treewidth at most $f(k)$?

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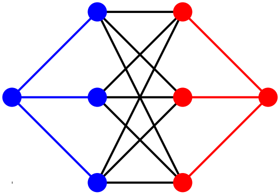

NO for the vertex-coverability question.

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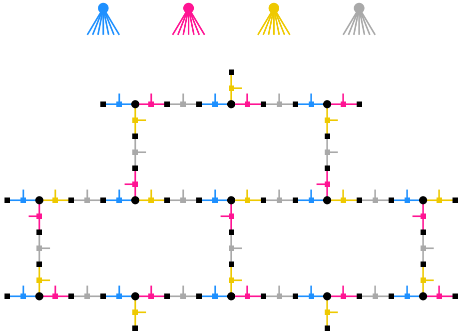

Source: Baside, Duron, Hodor, Liu, Nie 2025

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$k = 3$?

Thank you for your attention.