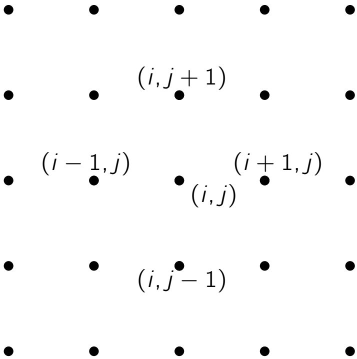
Structure of quasi-transitive graphs: planarity, minor exclusion and more.

Ugo Giocanti
Based on joint works with Louis Esperet and Clément Legrand-Duchesne.

Uniwersytet Jagielloński, Kraków

Séminaire Darboux, Montpellier.

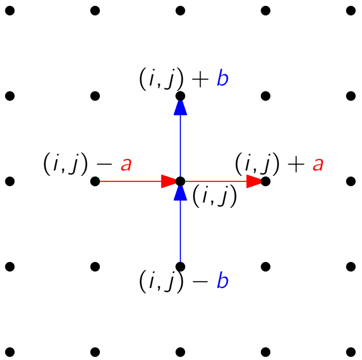
$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



$$(\Gamma, \cdot) := (\mathbb{Z}^2, +)$$

Cayley graphs

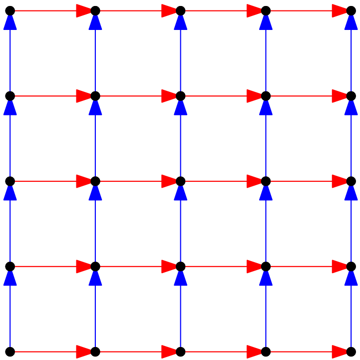
$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



$$(\Gamma, \cdot) := (\mathbb{Z}^2, +)$$
$$a := (1,0), b := (0,1)$$

Cayley graphs

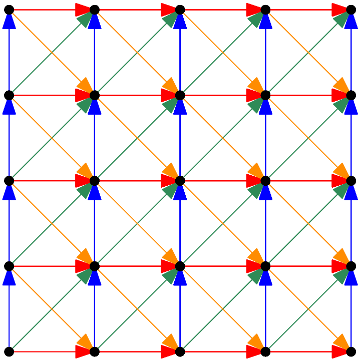
$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



$(\Gamma, \cdot) := (\mathbb{Z}^2, +)$
$a := (1, 0)$, $b := (0, 1)$

Cayley graphs

$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



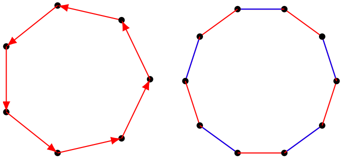
$(\Gamma, \cdot) := (\mathbb{Z}^2, +)$
$a := (1, 0)$, $b := (0, 1)$
$c := (1, 1)$, $d := (1, -1)$

Planar groups

- [Maschke 1896] Full list of all finite planar Cayley graphs.

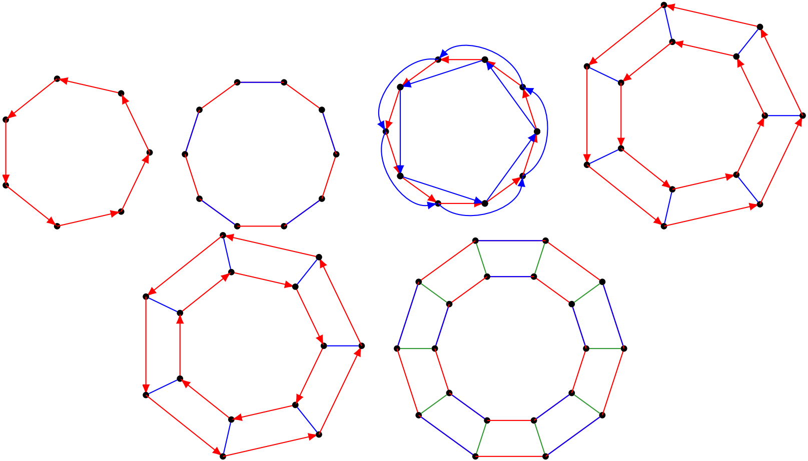
Planar groups

- [Maschke 1896] Full list of all finite planar Cayley graphs.



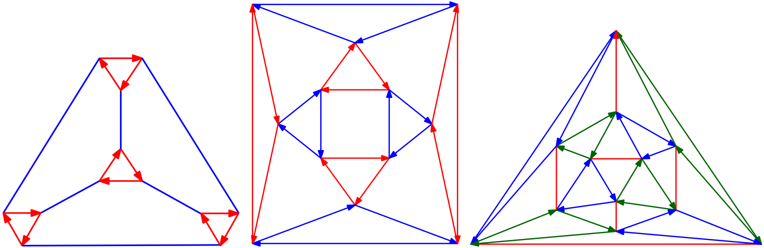
Planar groups

- [Maschke 1896] Full list of all finite planar Cayley graphs.



Planar groups

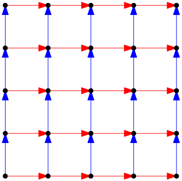
- [Maschke 1896] Full list of all finite planar Cayley graphs.

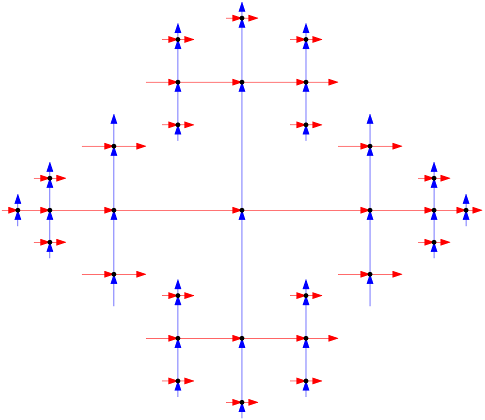


+ 15 others

Planar groups

- [Maschke 1896] Full list of all finite planar Cayley graphs.
- [Wilkie 1966], [MacBeath 1967], [Zieschang, Volgt, Coldeway 1980]: characterization of locally finite planar Cayley graphs with a vertex-accumulation-free embedding.





Planar groups

- [Maschke 1896] Full list of all finite planar Cayley graphs.
- [Wilkie 1966], [MacBeath 1967], [Zieschang, Volgt, Coldeway 1980]: characterization of locally finite planar Cayley graphs with a vertex-accumulation-free embedding.

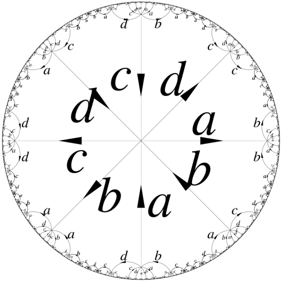


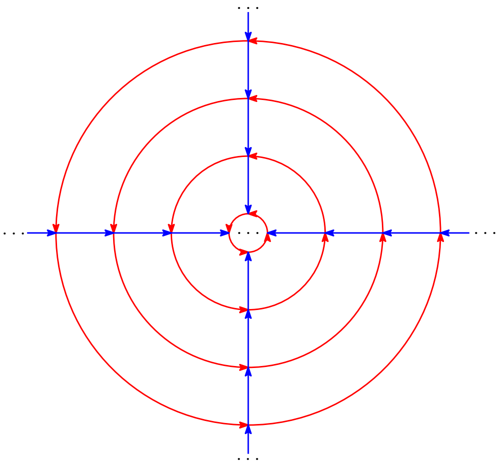
Image source: Yann Ollivier. A primer to geometric group theory.
`http://www.yann-ollivier.org/maths/primer.php`

Planar groups

- [Droms 2006] Decomposition method to construct all locally finite planar Cayley graphs.

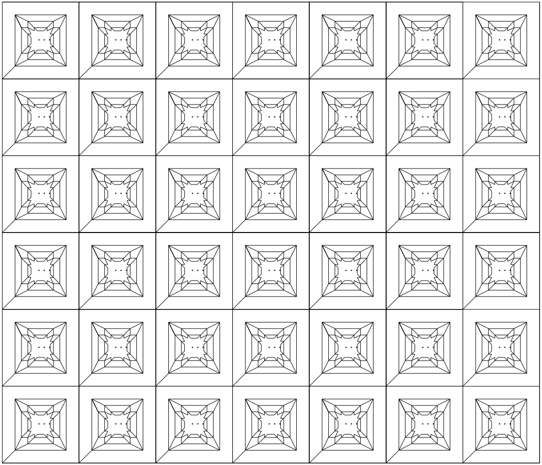
Planar groups

- [Droms 2006] Decomposition method to construct all locally finite planar Cayley graphs.



Planar groups

- [Droms 2006] Decomposition method to construct all locally finite planar Cayley graphs.



Quasi-transitive graphs

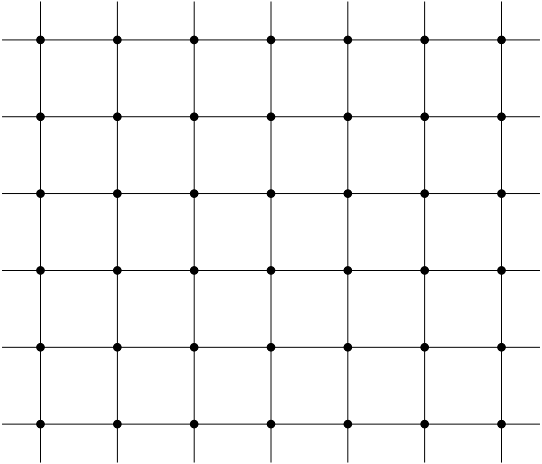
G: (connected) graph, countable vertex set, bounded degree.

Quasi-transitive graphs

G: (connected) graph, countable vertex set, bounded degree.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.

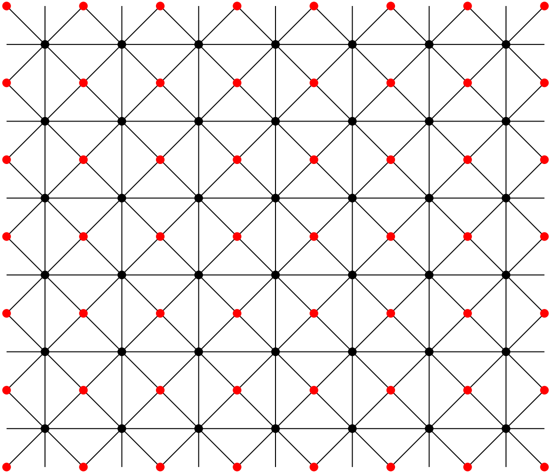
Quasi-transitive graphs

G: (connected) graph, countable vertex set, bounded degree.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.



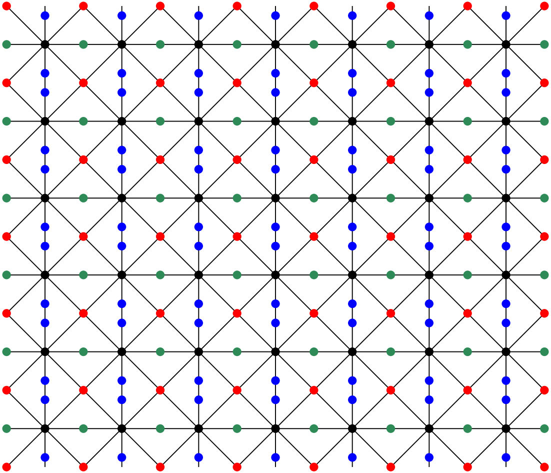
Quasi-transitive graphs

G: (connected) graph, countable vertex set, bounded degree.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.



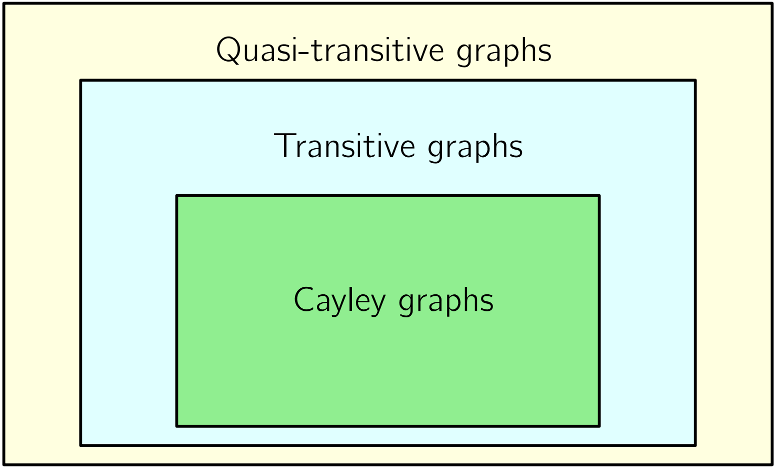
Quasi-transitive graphs

G: (connected) graph, countable vertex set, bounded degree.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.

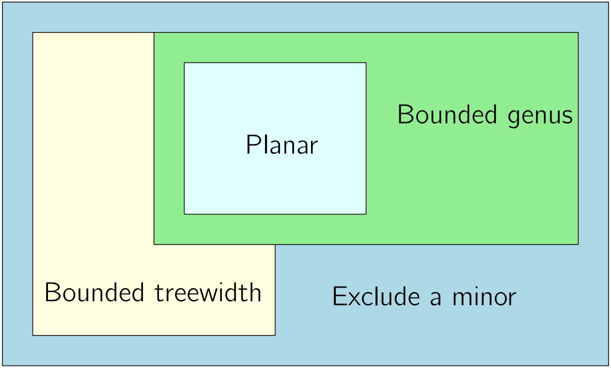


Quasi-transitive graphs

G: (connected) graph, countable vertex set, bounded degree.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.



In this presentation: structural characterizations of classes of quasi-transitive graphs defined by geometric properties.



Ends, accessibility

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.
$r \sim r'$ if for every finite $X \subseteq_{fin} V(G)$, the infinite components of r and r' are in the same connected component of $G \setminus X$.

Ends, accessibility

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \dots)$ in a graph G.
$r \sim r'$ if for every finite $X \subseteq_{fin} V(G)$, the infinite components of r and r' are in the same connected component of $G \setminus X$.
ends of G: equivalence classes of rays.

Ends, accessibility

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.
$r \sim r'$ if for every finite $X \subseteq_{fin} V(G)$, the infinite components of r and r' are in the same connected component of $G \setminus X$.
ends of G: equivalence classes of rays.

Theorem (Hopf '43, Freudenthal, '44)

A Cayley graph has either 0, 1, 2 *or infinitely many ends.*

Ends ω, ω' are k-distinguishable if there is a set $X \subseteq \Gamma$ of size at most k separating their rays.

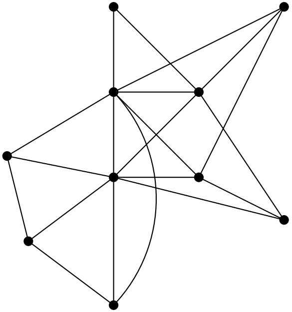
Ends, accessibility

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.
$r \sim r'$ if for every finite $X \subseteq_{fin} V(G)$, the infinite components of r and r' are in the same connected component of $G \setminus X$.
ends of G: equivalence classes of rays.

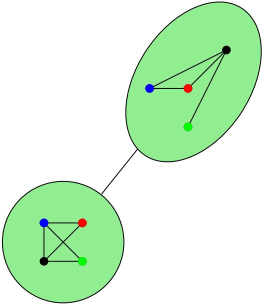
Theorem (Hopf '43, Freudenthal, '44)

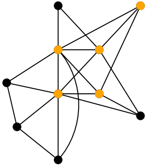
A Cayley graph has either 0, 1, 2 *or infinitely many ends.*

Ends ω, ω' are k-distinguishable if there is a set $X \subseteq \Gamma$ of size at most k separating their rays.
G is accessible if there is some $k \geqslant 0$ such that all its ends are k-distinguishable.

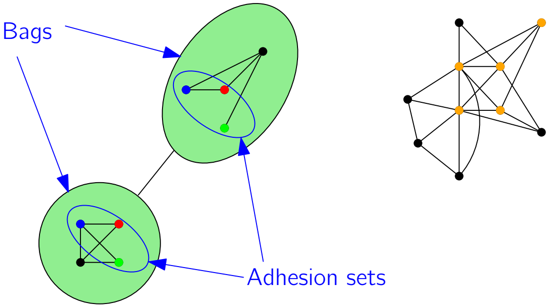


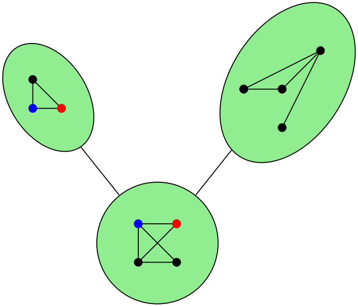
Tree-decompositions

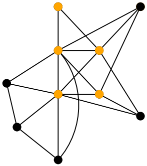




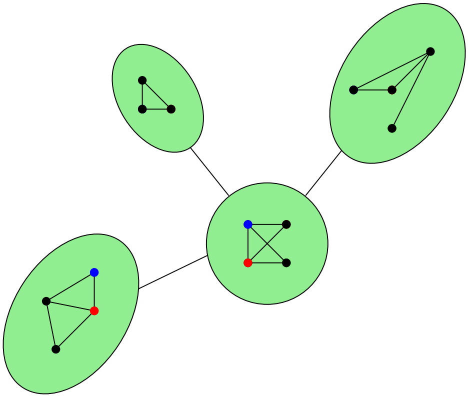
Tree-decompositions

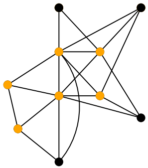




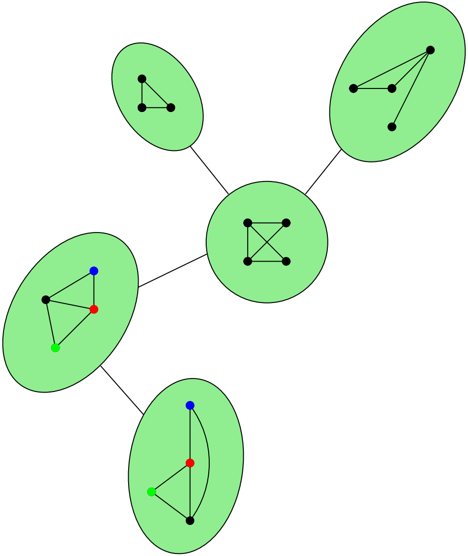


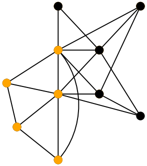
Tree-decompositions



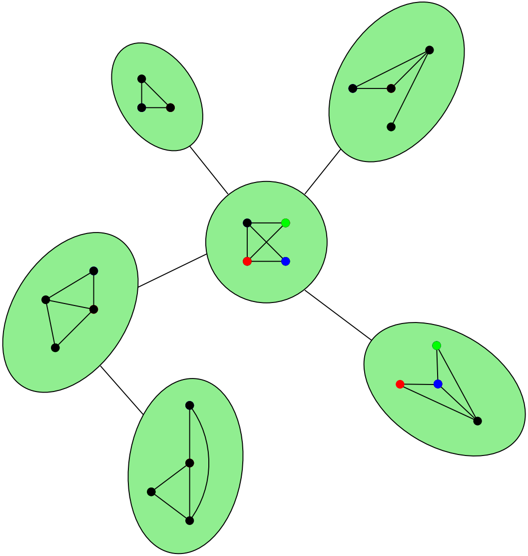


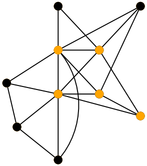
Tree-decompositions





Tree-decompositions





Tree-decompositions

Tree decomposition of G: pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is the collection of bags.

Tree-decompositions

Tree decomposition of G: pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is the collection of bags.
$(T, \mathcal{V})$ is canonical if $\mathrm{Aut}(G)$ induces an action on T such that for every $t \in V(T)$ and $\gamma \in \mathrm{Aut}(G)$, $V_{\gamma \cdot t} = \gamma \cdot V_t$.

Tree-decompositions

Tree decomposition of G: pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is the collection of bags.

$(T, \mathcal{V})$ is canonical if $\mathrm{Aut}(G)$ induces an action on T such that for every $t \in V(T)$ and $\gamma \in \mathrm{Aut}(G)$, $V_{\gamma \cdot t} = \gamma \cdot V_t$.

Canonical tree-decompositions $\approx$ Bass-Serre splittings (Hamann, Lehner, Miraftab, Rühmann 2022).

Tree-decompositions

Tree decomposition of G: pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is the collection of bags.

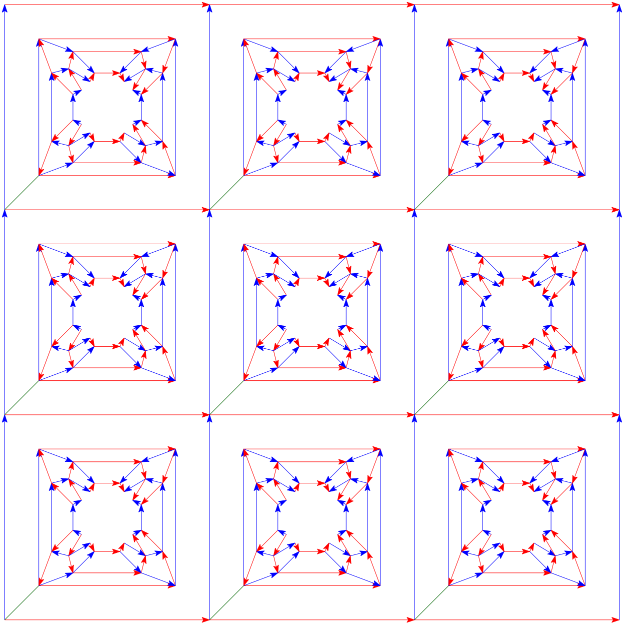
$(T, \mathcal{V})$ is canonical if $\mathrm{Aut}(G)$ induces an action on T such that for every $t \in V(T)$ and $\gamma \in \mathrm{Aut}(G)$, $V_{\gamma \cdot t} = \gamma \cdot V_t$.

Canonical tree-decompositions ≈ Bass-Serre splittings (Hamann, Lehner, Miraftab, Rühmann 2022).

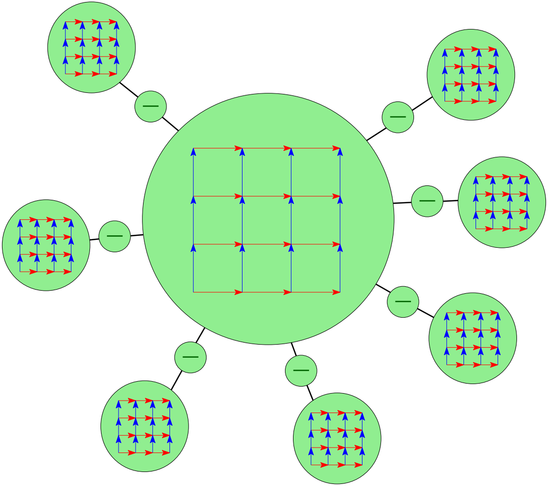
Theorem (Thomassen, Woess 1991, Carmesin, Hamann, Miraftab 2022)

A locally finite quasi-transitive graph is accessible if and only if it admits a (canonical) tree-decomposition of finite adhesion whose parts $G[V_t]$ are either finite or one-ended.

Planar quasi-transitive graphs



Planar quasi-transitive graphs



Planar quasi-transitive graphs

Analogous of Droms' decomposition in quasi-transitive graphs.

Theorem (G. 2025, built on Hamann 2018)

Every planar locally finite 3*-connected quasi-transitive graph* G *admits a canonical tree-decomposition whose edge-separations correspond to cycle-separations in the (unique) embedding of* G*, and where every part is a quasi- transitive subgraph of* G *admitting a vertex-accumulation-free planar embedding.*

Minors

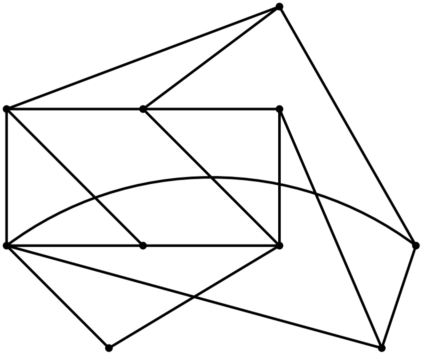
A graph H is a minor of G if H can be obtained from G after performing the following operations:

- vertex deletions;
- edge deletions;
- edge contractions.

Minors

A graph H is a minor of G if H can be obtained from G after performing the following operations:

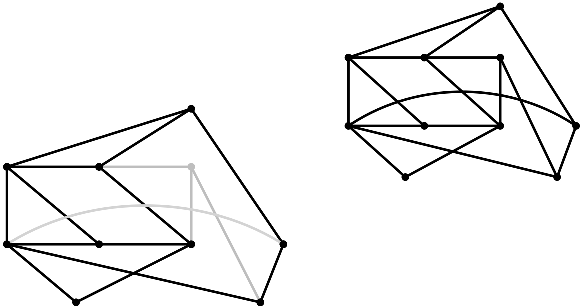
- vertex deletions;
- edge deletions;
- edge contractions.



Minors

A graph H is a minor of G if H can be obtained from G after performing the following operations:

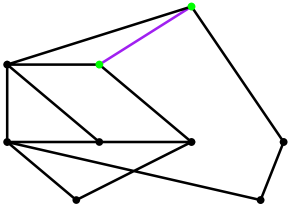
- vertex deletions;
- edge deletions;
- edge contractions.

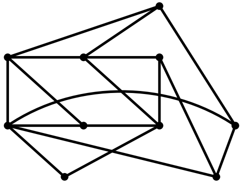


Minors

A graph H is a minor of G if H can be obtained from G after performing the following operations:

- vertex deletions;
- edge deletions;
- edge contractions.

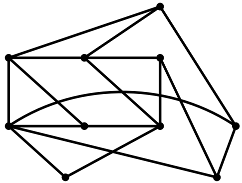


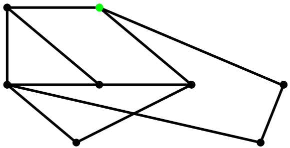


Minors

A graph H is a minor of G if H can be obtained from G after performing the following operations:

- vertex deletions;
- edge deletions;
- edge contractions.

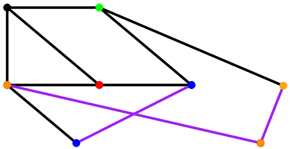


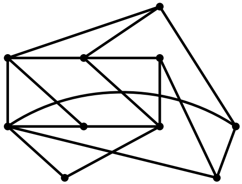


Minors

A graph H is a minor of G if H can be obtained from G after performing the following operations:

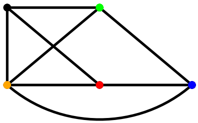
- vertex deletions;
- edge deletions;
- edge contractions.

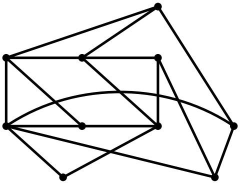




A graph H is a minor of G if H can be obtained from G after performing the following operations:

- vertex deletions;
- edge deletions;
- edge contractions.





Graph Minor Structure Theorem

- [Robertson, Seymour 2003] "If a finite graph G excludes some fixed minor H, then G has a tree-decomposition where each torso almost embeds in a surface of bounded genus."

Graph Minor Structure Theorem

- [Robertson, Seymour 2003] "If a finite graph G excludes some fixed minor H, then G has a tree-decomposition where each torso almost embeds in a surface of bounded genus."
- [Kříž, Thomas 1990] "Extends to infinite graphs excluding some finite minor."

Graph Minor Structure Theorem

- [Robertson, Seymour 2003] "If a finite graph G excludes some fixed minor H, then G has a tree-decomposition where each torso almost embeds in a surface of bounded genus."
- [Kříž, Thomas 1990] "Extends to infinite graphs excluding some finite minor."
- [Diestel, Thomas 1999] "A similar result for graphs excluding the countable clique K_∞ as a minor."

Graph Minor Structure Theorem

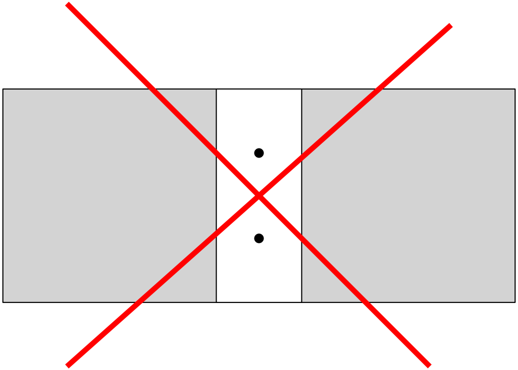
- [Robertson, Seymour 2003] "If a finite graph G excludes some fixed minor H, then G has a tree-decomposition where each torso almost embeds in a surface of bounded genus."
- [Kříž, Thomas 1990] "Extends to infinite graphs excluding some finite minor."
- [Diestel, Thomas 1999] "A similar result for graphs excluding the countable clique K_∞ as a minor."

→ None of these results are canonical.

Minors in quasi-4-connected graphs

G is quasi-4-connected if:

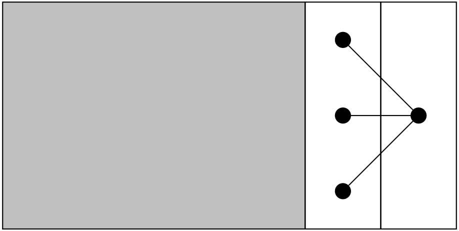
- G is 3-connected;
- the only separations of order 3 in G are between a single vertex and the remainder of the graph.



Minors in quasi-4-connected graphs

G is quasi-4-connected if:

- G is 3-connected;
- the only separations of order 3 in G are between a single vertex and the remainder of the graph.



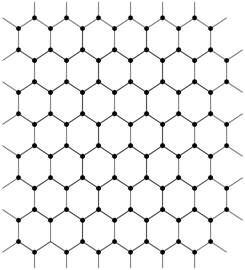
Minors in quasi-4-connected graphs

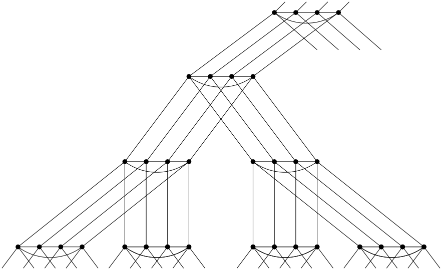
G is quasi-4-connected if:

- G is 3-connected;
- the only separations of order 3 in G are between a single vertex and the remainder of the graph.

Theorem (Thomassen 1992)

If G is locally finite, quasi-transitive, quasi-4-connected and excludes K_∞ as a minor, then G is either planar or it has finite treewidth.





Minors in quasi-4-connected graphs

G is quasi-4-connected if:

- G is 3-connected;
- the only separations of order 3 in G are between a single vertex and the remainder of the graph.

Theorem (Thomassen 1992)

If G is locally finite, quasi-transitive, quasi-4-connected and excludes K_∞ as a minor, then G is either planar or it has finite treewidth.

Corollary (Thomassen 1992)

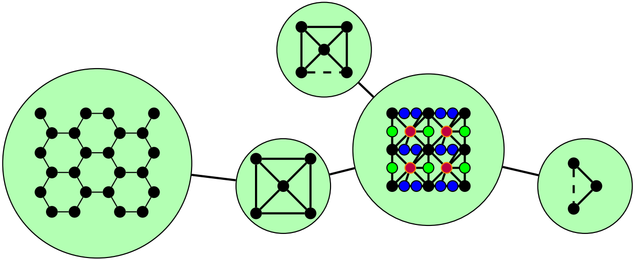
If G is locally finite, quasi-4-connected and quasi-transitive, and if G has every finite graph as a minor, then G has K_∞ as a minor.

→ Question (Thomassen 1992): Can we drop the quasi-4-connectivity condition?

Minor excluded quasi-transitive graphs

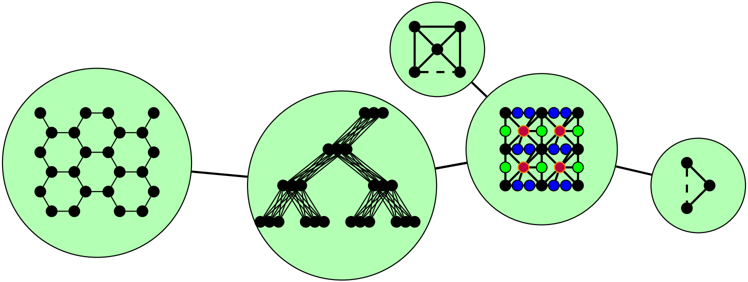
Theorem (Esperet, G., Legrand-Duchesne 2023 (finite/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



Theorem (Esperet, G., Legrand-Duchesne 2023 (finite treewidth/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.



Minor excluded quasi-transitive graphs

Theorem (Esperet, G., Legrand-Duchesne 2023 (finite treewidth/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.

Corollary

If G is locally finite, quasi-transitive and has every finite graph as a minor, then it also has K_∞ as a minor.

Minor excluded quasi-transitive graphs

Theorem (Esperet, G., Legrand-Duchesne 2023 (finite treewidth/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.

Corollary

If G is locally finite, quasi-transitive and has every finite graph as a minor, then it also has K_∞ as a minor.

Proof based on results and methods from [Grohe '16] and [Carmesin, Hamann, Miraftab '22].

Minor excluded quasi-transitive graphs

Theorem (Esperet, G., Legrand-Duchesne 2023 (finite treewidth/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.

Theorem (MacManus 2023)

Let G be a quasi-transitive locally finite graph quasi-isometric to a planar graph. Then there G admits a canonical tree-decomposition $(T, \mathcal{V})$, of finite adhesion whose torsos are quasi-transitive finite or quasi-isometric to some complete Riemannian surface.

Minors and quasi-isometries to planar graphs

Theorem (Esperet, G., 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar graph of bounded degree.

Minors and quasi-isometries to planar graphs

Theorem (Esperet, G., 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar graph of bounded degree.

Theorem (Hamann, 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar quasi-transitive graph of bounded degree.

Minors and quasi-isometries to planar graphs

Theorem (Esperet, G., 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar graph of bounded degree.

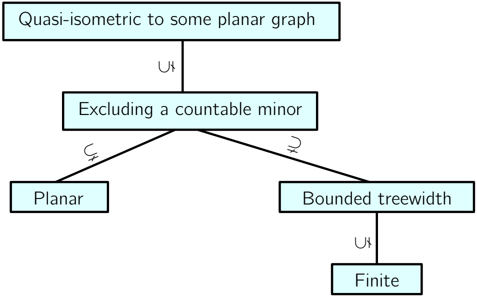
Theorem (Hamann, 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar quasi-transitive graph of bounded degree.

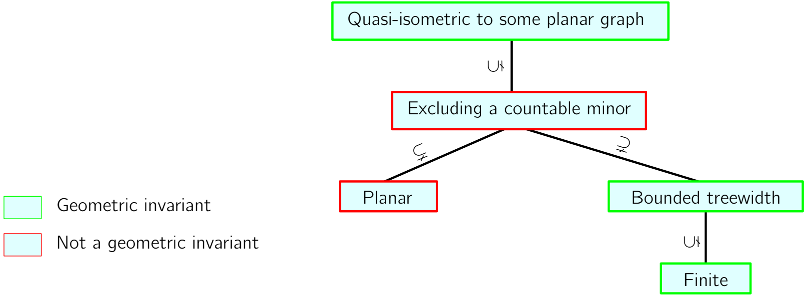
Theorem (MacManus, 2024)

Every bounded degree quasi-transitive graph excluding a minor is quasi-isometric to some planar Cayley graph of bounded degree.

Minors and quasi-isometries to planar graphs



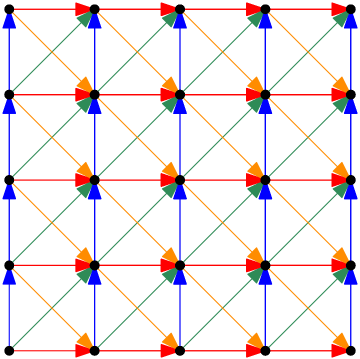
Minors and quasi-isometries to planar graphs



Beyond planarity

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.



Beyond planarity

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.

Theorem (Esperet, G. 2024)

The property of being k-planar for some $k \in \mathbb{N}$ is a geometric invariant for bounded degree graphs.

Beyond planarity

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.

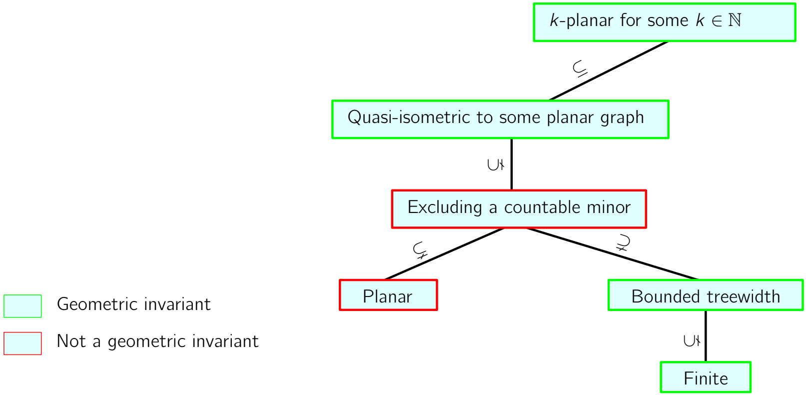
Theorem (Esperet, G. 2024)

The property of being k-planar for some $k \in \mathbb{N}$ is a geometric invariant for bounded degree graphs.

Corollary

Every graph of bounded degree which is quasi-isometric to a planar graph is k-planar for some $k \in \mathbb{N}$.

Beyond planarity



Beyond planarity

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.

Theorem (Esperet, G. 2024)

The property of being k-planar for some $k \in \mathbb{N}$ is a geometric invariant for bounded degree graphs.

Corollary

Every graph of bounded degree which is quasi-isometric to a planar graph is k-planar for some $k \in \mathbb{N}$.

Beyond planarity

G is k-planar ($k \in \mathbb{N}$) if it has an drawing in $\mathbb{R}^2$ such that each edge is crossed by at most k other edges.

Theorem (Esperet, G. 2024)

The property of being k-planar for some $k \in \mathbb{N}$ is a geometric invariant for bounded degree graphs.

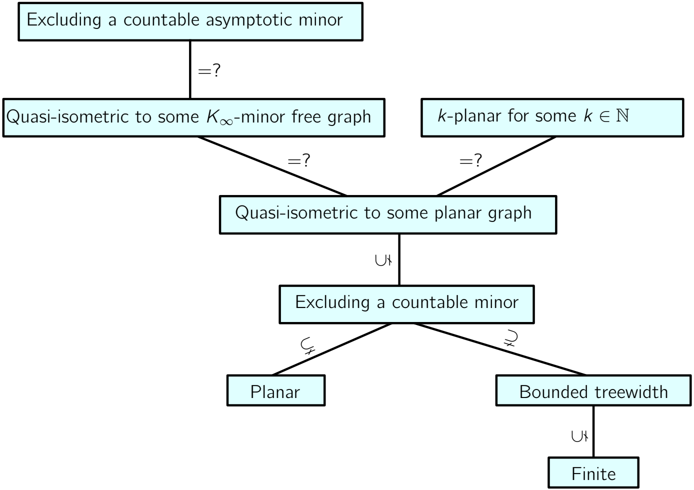
Corollary

Every graph of bounded degree which is quasi-isometric to a planar graph is k-planar for some $k \in \mathbb{N}$.

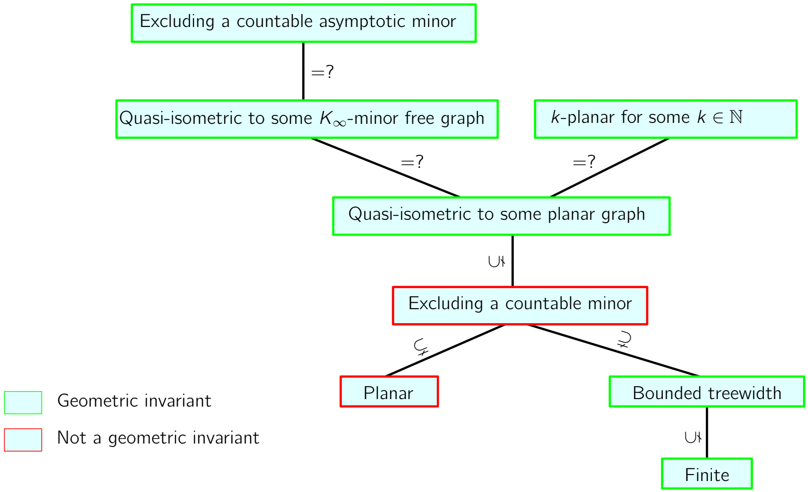
Conjecture (Georgakopoulos, Papasoglou 2023)

Let G be a bounded degree quasi-transitive graph. Then G is quasi-isometric to a planar graph if and only if it is k-planar for some $k \in \mathbb{N}$.

Beyond planarity



Beyond planarity



Thank you for your attention.