The structure of quasi-transitive graphs avoiding a minor with applications to the Domino Conjecture.

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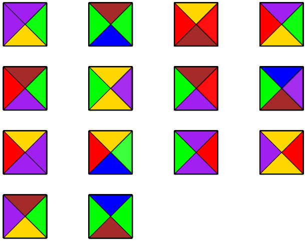
Wang tiling problem



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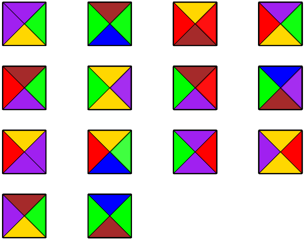

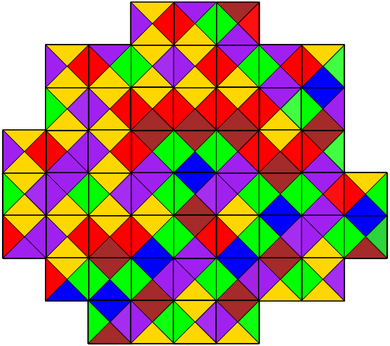

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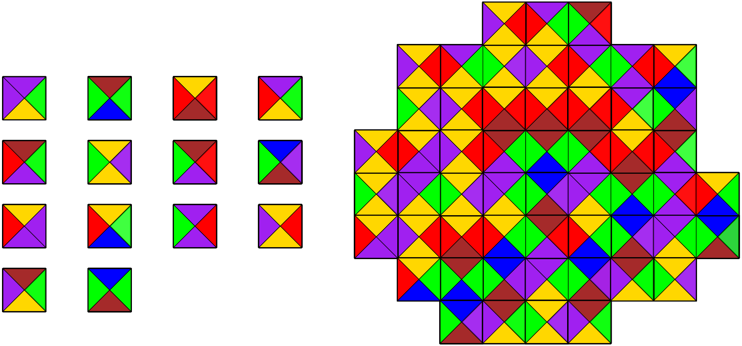
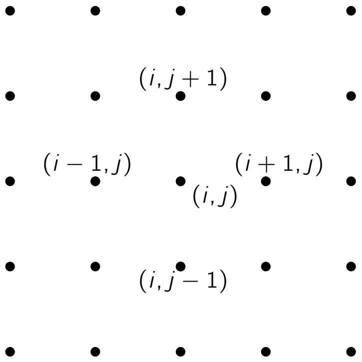


Theorem (Berger, '66)

The Wang tiling problem is undecidable.

$(\Gamma, \cdot)$: group, S: finite set of generators. $\mathrm{Cay}(\Gamma, S)$: graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



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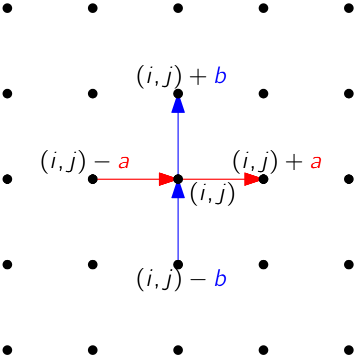

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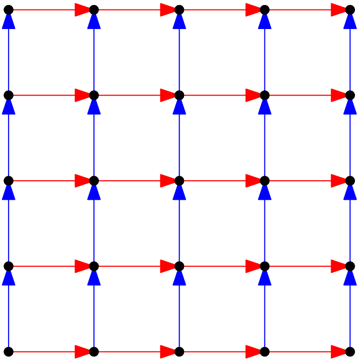

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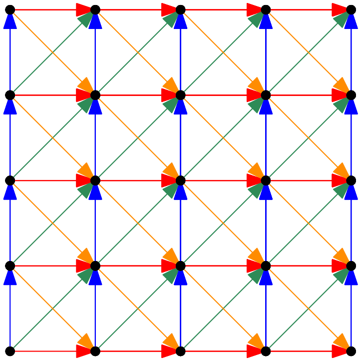

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Domino Problem on groups

Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

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Domino problem on (Γ, S):
Input: a finite alphabet Σ and a finite set $\mathcal{F} = \{p_1, \ldots, p_t\}$ of forbidden patterns.
Question: Is there a coloring $c : V(G) \to \Sigma$ avoiding $\mathcal{F}$?

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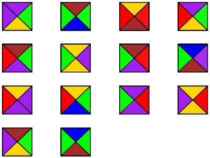
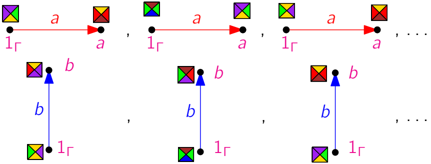
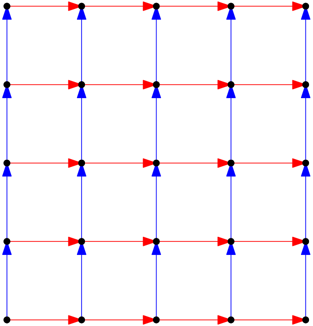


Local rules:





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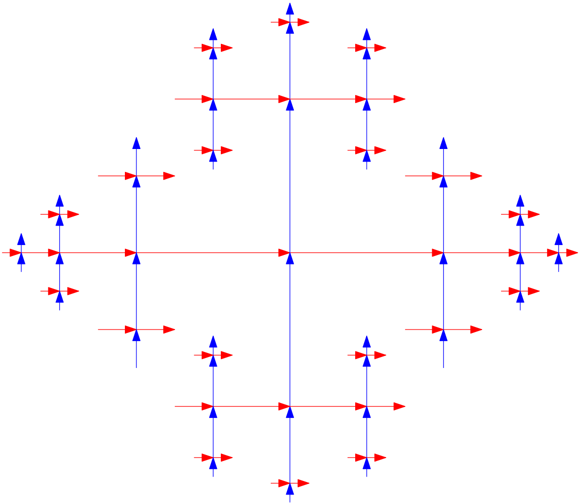
(Virtually-)Free groups

Free-groups: groups Γ that admit a tree as a Cayley graph.



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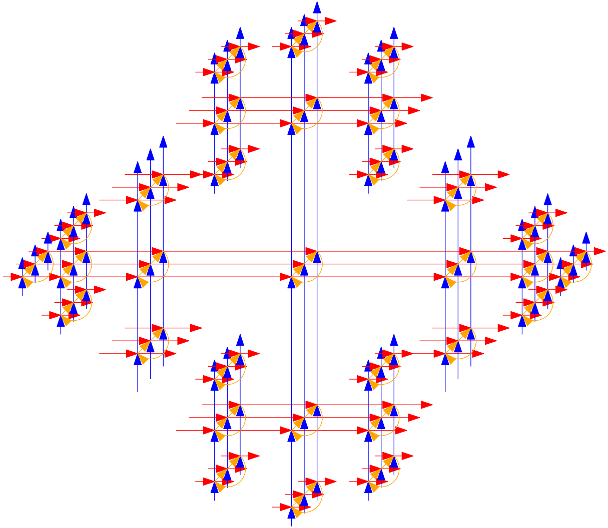
Γ is virtually-free if it admits some subgroup of finite index which is free.



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Theorem (Karass, Pietrowski, Solitar '73)

Γ is virtually-free if and only if one/all its Cayley graphs have bounded treewidth.

Claim: If G has bounded degree, then G has bounded treewidth if and only if G is a subgraph of a k-blow up of a tree for some $k \geqslant 0$.

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Our results: true when Γ is planar, and more generally minor-excluded.

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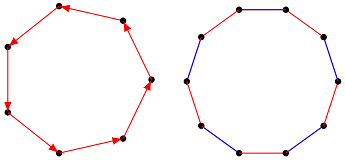

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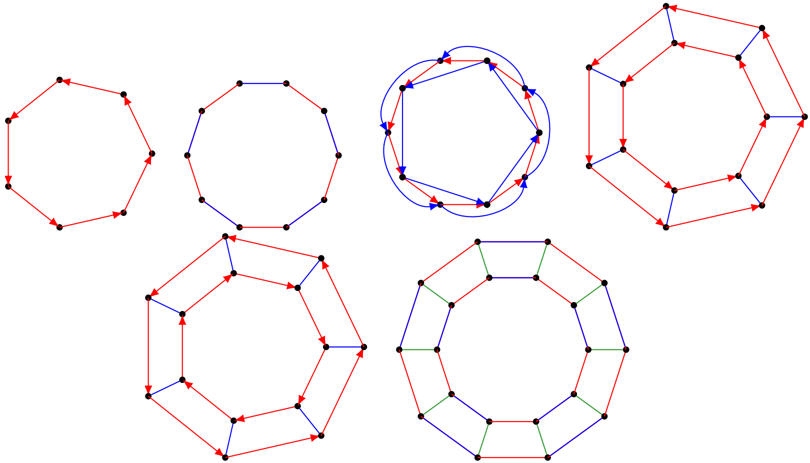

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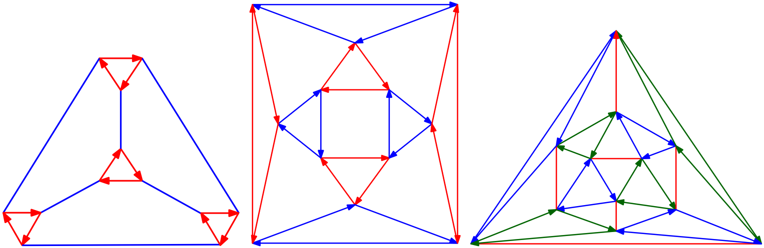

+ 15 others

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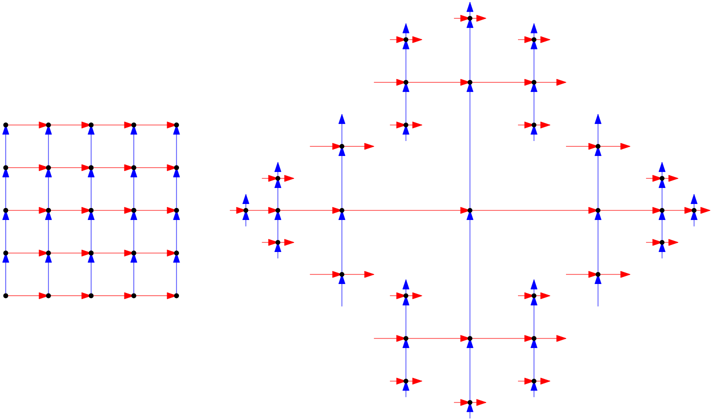

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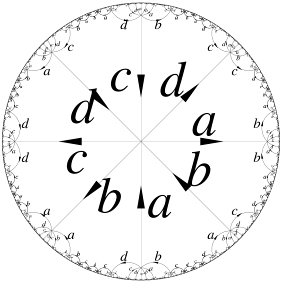

Image source: Yann Ollivier. A primer to geometric group theory.
`http://www.yann-ollivier.org/maths/primer.php`

Planar groups

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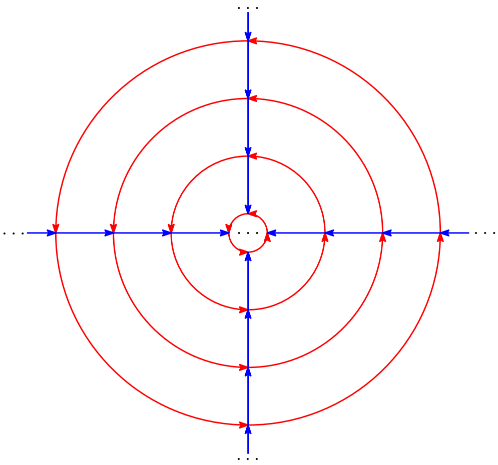

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Decidable on virtually-free groups;

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Theorem

The conjecture is true for planar groups.

G: infinite graph.
Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.

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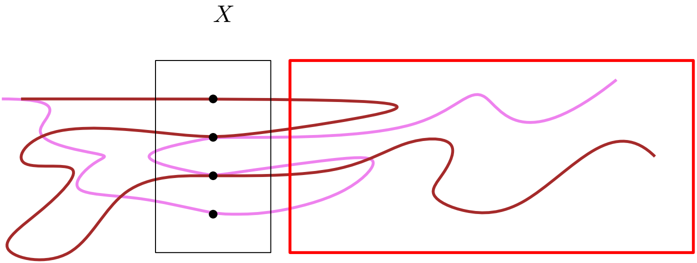

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Ends of G: equivalence classes of rays.

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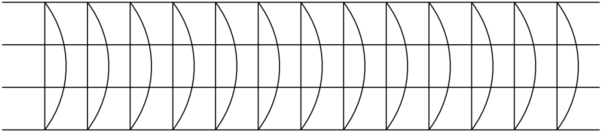

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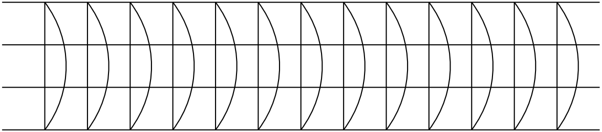

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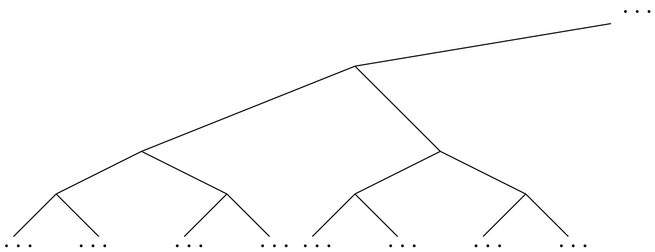

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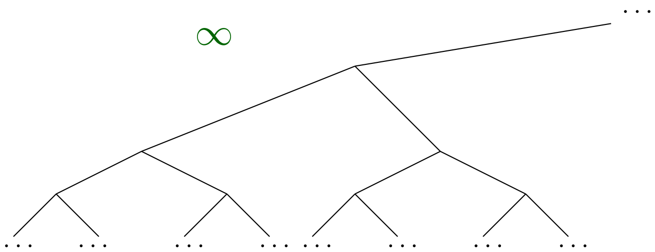

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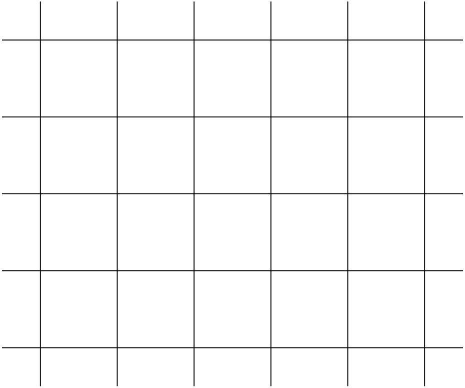

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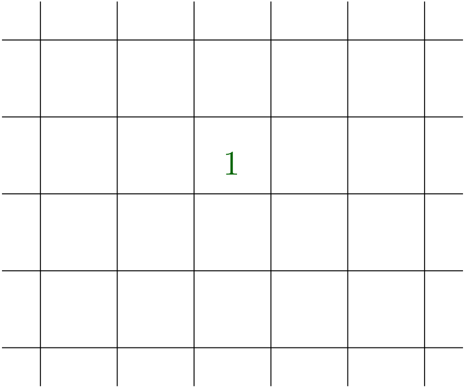

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How many ends in the following graphs?

Theorem (Hopf '43, Freudenthal, '44)

A Cayley graph has either 0, 1, 2 or infinitely many ends.

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If a group Γ is planar with one end, then it contains the fundamental group of a surface as a subgroup of finite index.

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If a group Γ is planar with one end, then it contains the fundamental group of a surface as a subgroup of finite index.

In particular, in such groups the Domino problem is undecidable.

Accessibility

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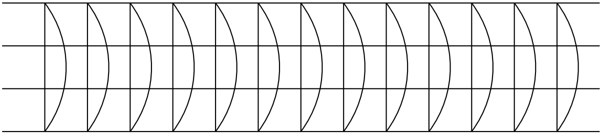

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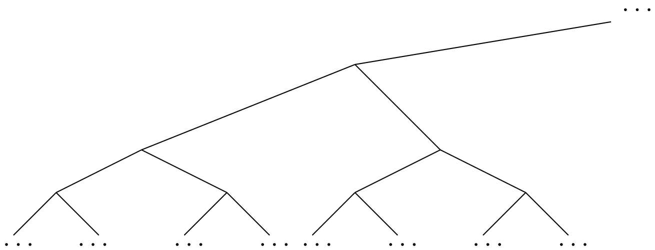

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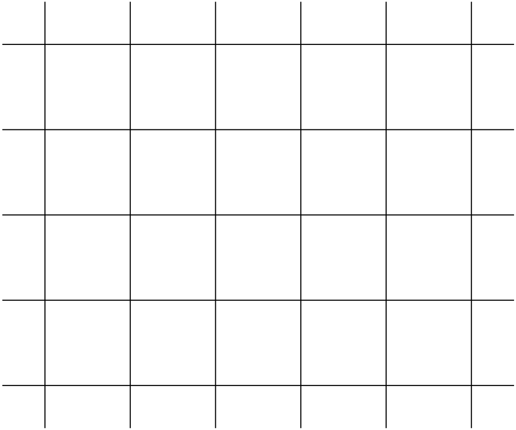

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Theorem (Bass-Serre theory)

If Γ is accessible, then

- *either Γ is virtually free*
- *or Γ contains a finitely generated subgroup with one end.*

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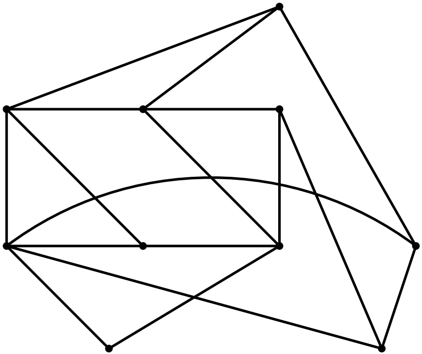

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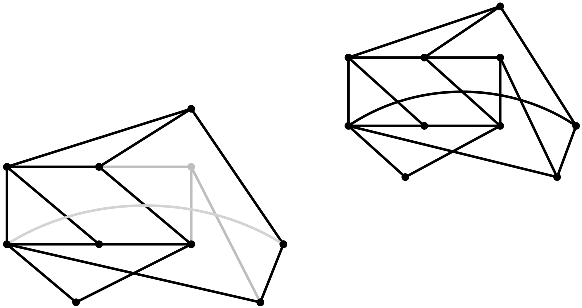

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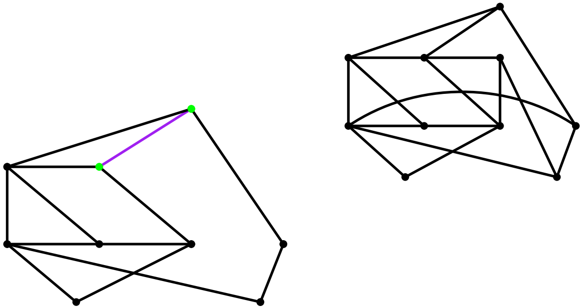

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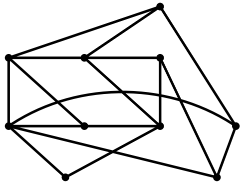

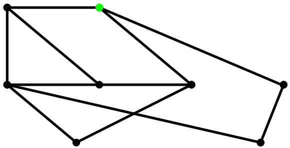

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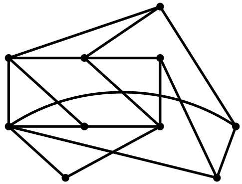

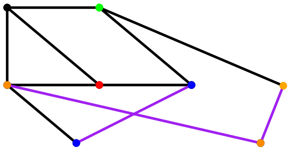

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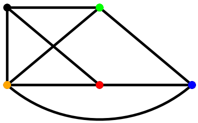

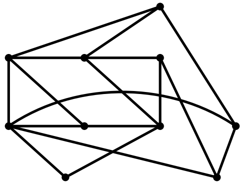

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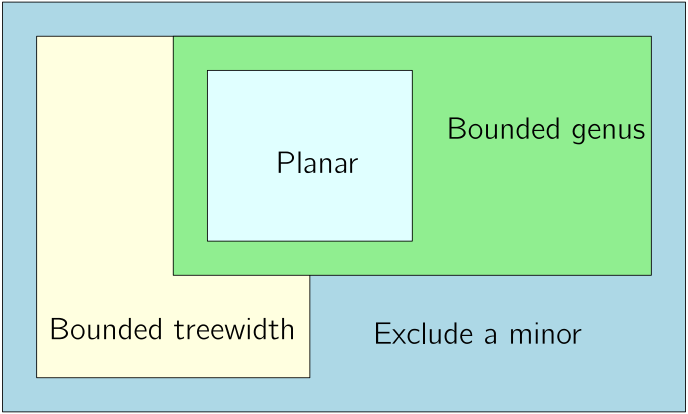

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Remark: G minor-excluded $\Leftrightarrow$ G is K_∞-minor free.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.
A group is minor excluded if one of its Cayley graphs excludes a (countable) minor.
Remark: G minor-excluded $\Leftrightarrow$ G is K_∞-minor free.

Theorem (Esperet, G., Legrand-Duchesne, 2024)

The Domino conjecture is true for planar groups and more generally for minor-excluding groups.

Quasi-transitive graphs

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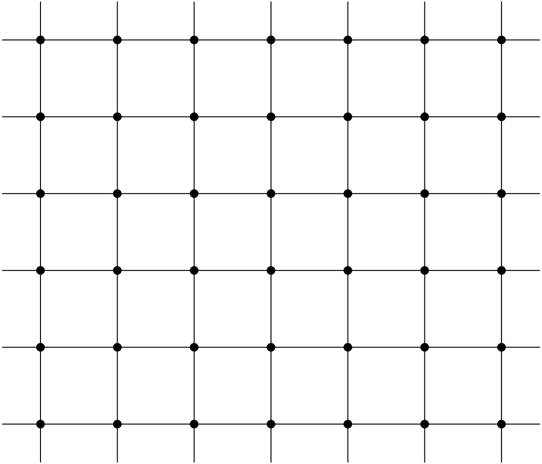

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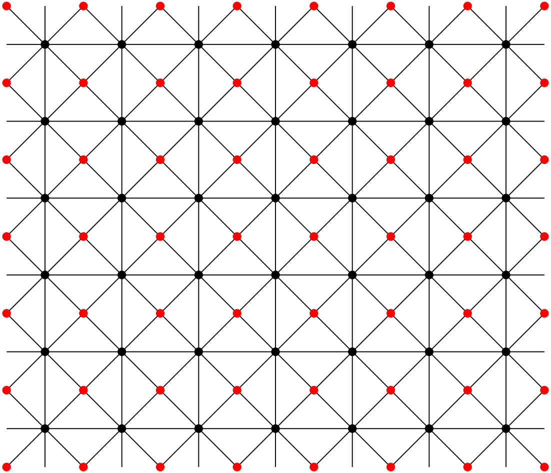

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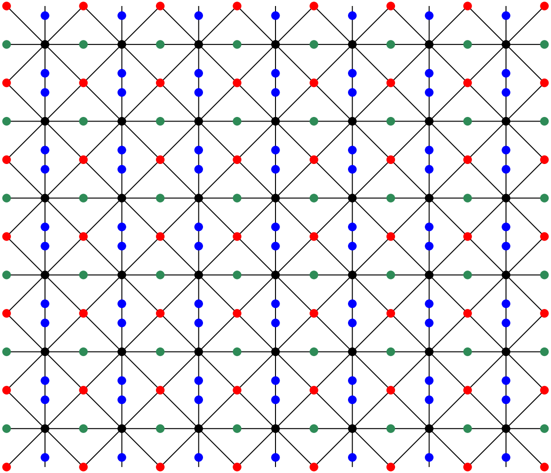

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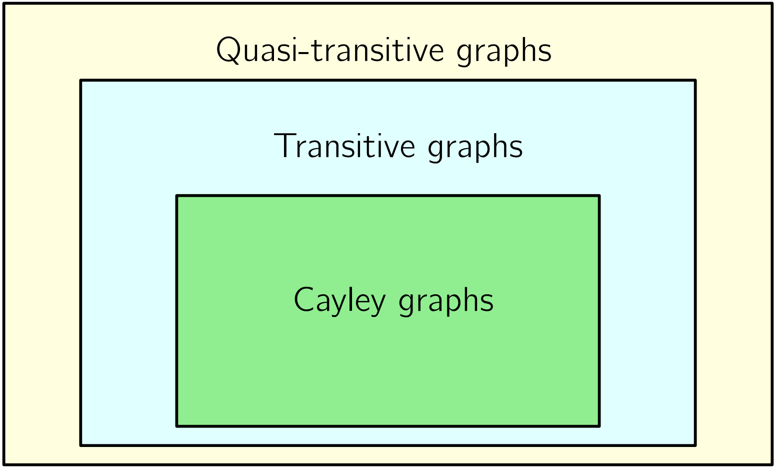
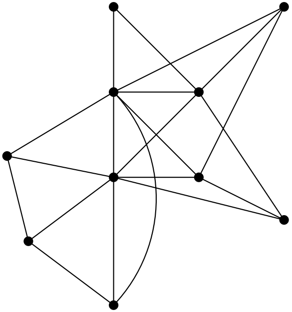


Tree-decompositions



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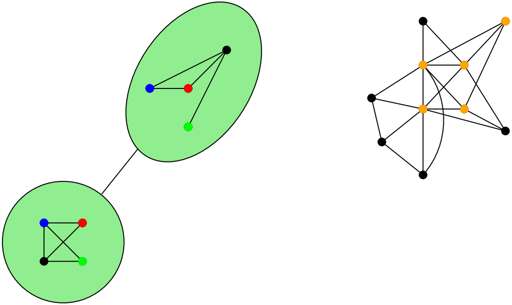

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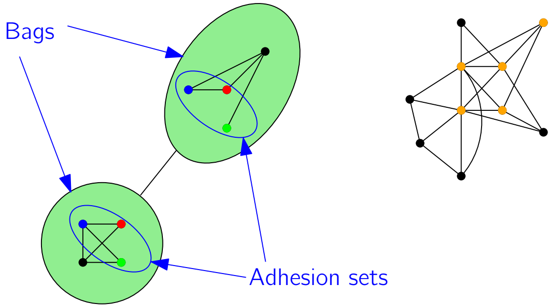

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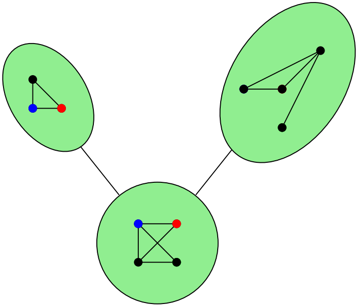

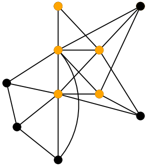

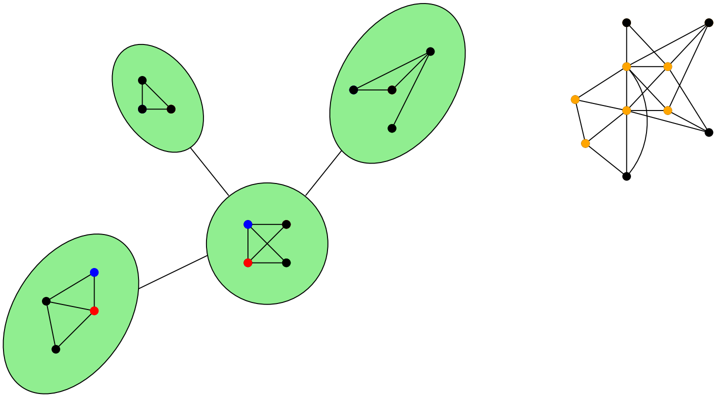

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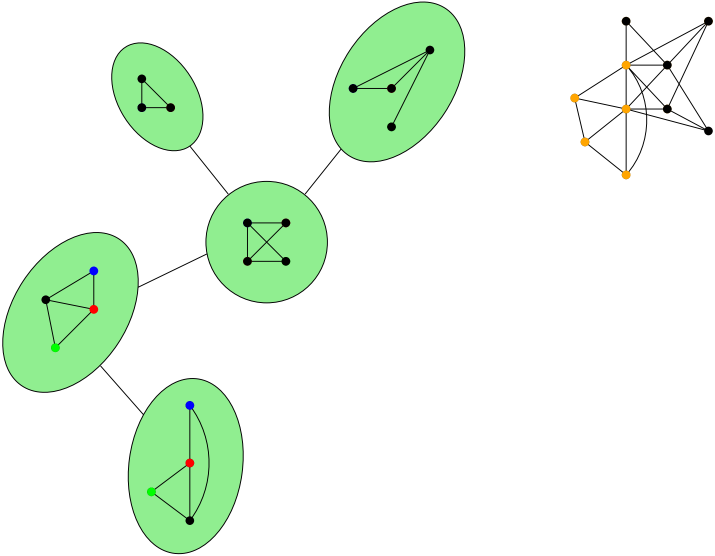
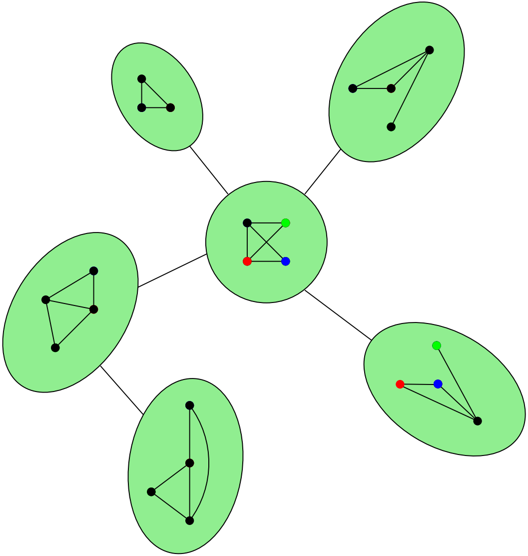
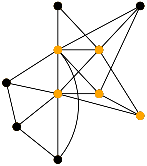
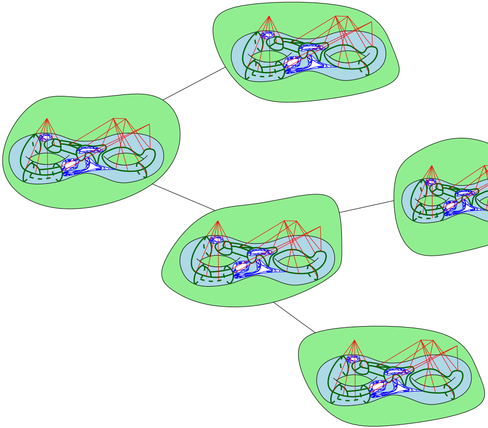






Graph Minor Structure Theorem

- [Robertson, Seymour 2003] "If a finite graph G excludes some fixed minor H, then G has a tree-decomposition where each torso almost embeds in a surface of bounded genus."



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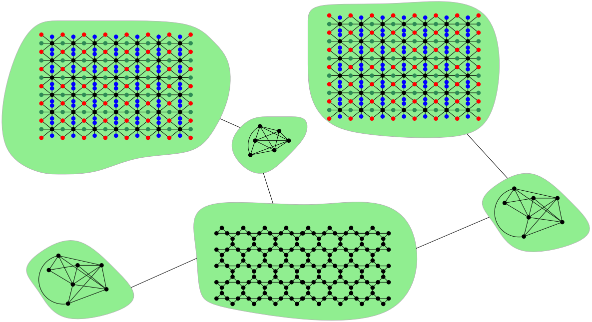
→ None of these results are canonical.

Minor excluded quasi-transitive graphs

Theorem (Esperet, G., Legrand-Duchesne 2023 (finite/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



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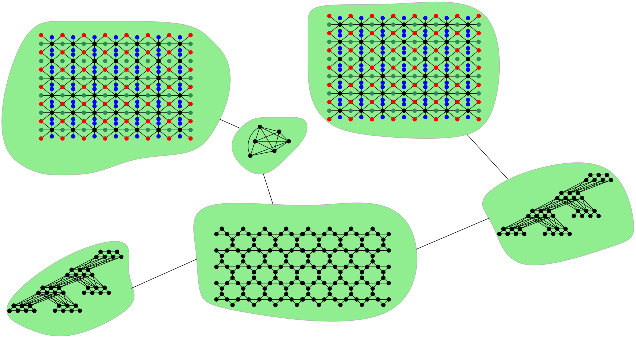
Theorem (Esperet, G., Legrand-Duchesne 2023 (finite treewidth/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.



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Corollary

If G is locally finite, quasi-transitive and has every finite graph as a minor, then it also has K_∞ as a minor.

Answers a question of Thomassen (1992), who proved it for quasi-4-connected graphs.

Application: Finite presentability, Accessibility.

Theorem (Droms '06)

Planar groups are finitely presented.

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Every minor-excluding finitely generated group Γ is finitely presented.

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Every minor-excluding finitely generated group Γ is accessible.

Combined with a result of Babai (1977), and Bass-Serre theory, it implies:

Corollary

If Γ is finitely generated, minor excluded, then

- *either Γ is virtually-free,*
- *or Γ has a planar one-ended subgroup.*

Conclusion

- Prove results on groups by working in the more general world of quasi-transitive graphs.
- Key tool: canonicity (allows to do induction in the context of tree-decompositions).

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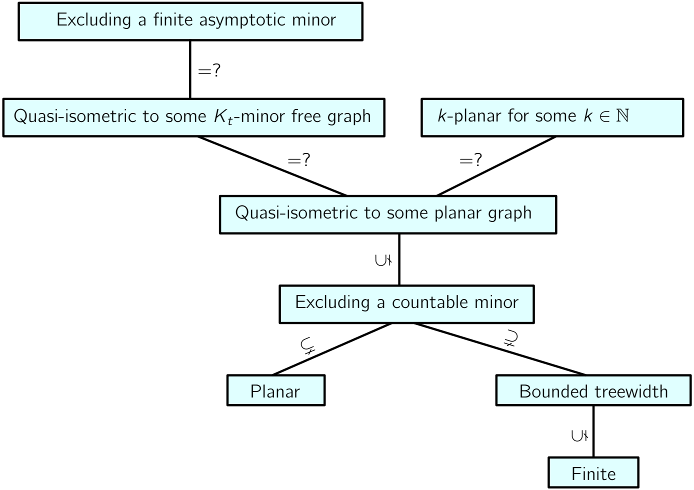
Questions:

- Analogous conjecture of Caroll and Penland (2015) on aperiodicity of tilings.
- Larger classes? Could Coarse Graph Theory help?

Conclusion


Excluding a finite asymptotic minor
=?
Quasi-isometric to some K_t-minor free graph
k-planar for some $k \in \mathbb{N}$
=?
=?
Quasi-isometric to some planar graph
$\supsetneq$
Excluding a countable minor
$\subsetneq$
$\supsetneq$
Planar
Bounded treewidth
$\supsetneq$
Finite

Conclusion

Dziękuję bardzo!