Cycle bases with low congestion in minor-excluded graphs

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Joint work with Colin Geniet[2]

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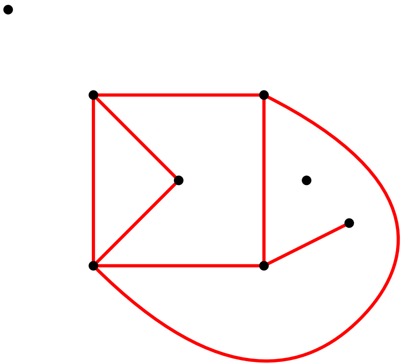
Cycle basis

Given G_1, G_2, their $\mathbb{F}_2$-sum $G_1 \oplus G_2$ is the graph $(V(G_1) \cup V(G_2), E(G_1)\Delta E(G_2))$.



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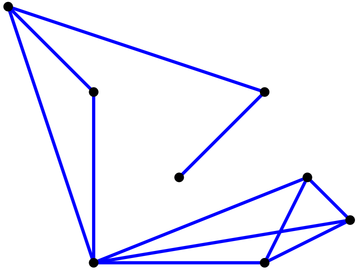

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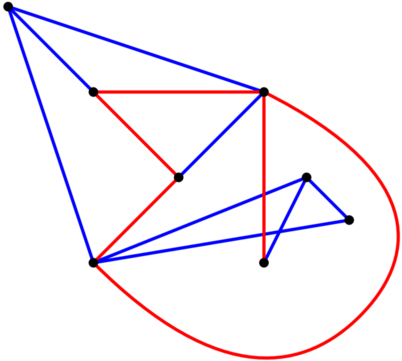

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A graph is even if all its vertices have even degree.
The cycle space $\mathcal{C}(G)$ is the set of all even subgraphs of G (equiped with $\oplus$).

Remark

The set of cycles of a graph generates its cycle space.

A cycle basis of G is a set of cycles generating $\mathcal{C}(G)$.

MacLane's planarity criterion

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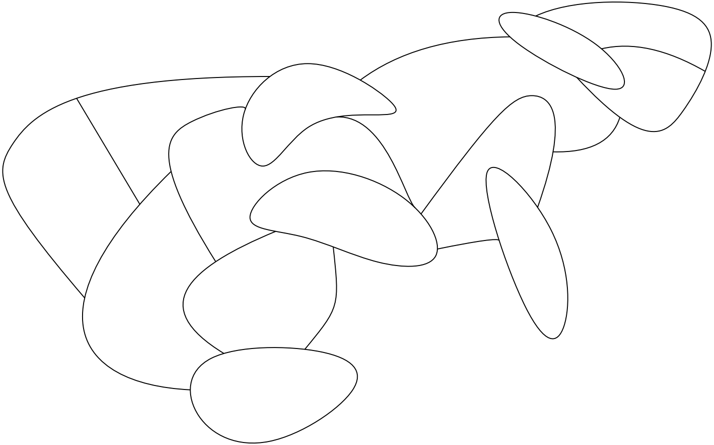
The set of facial cycles $\mathcal{C}_{\text{plan}}$ of a 2-connected plane graph forms a cycle basis.



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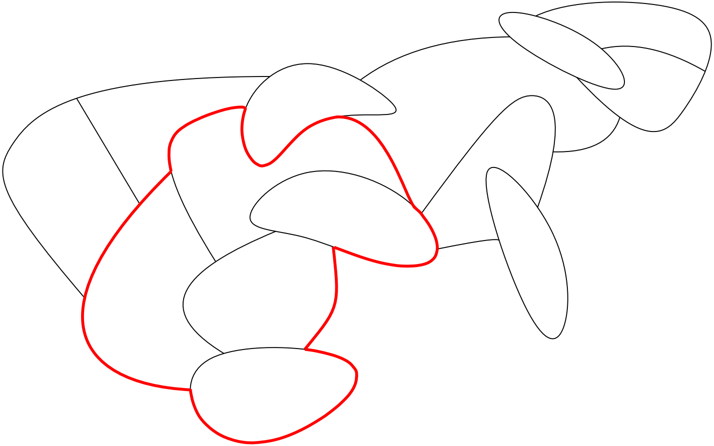

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The edge-congestion of a cycle basis is the minimum $k \geqslant 0$ such that each edge of G appears in at most k elements of $\mathcal{C}$.
The basis-number $\text{bn}(G)$ of G is the minimum k such that G has a cycle-basis with edge-congestion k.

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Theorem (MacLane's planarity criterion (1937))

A graph G is planar if and only if $\text{bn}(G) \leqslant 2$.

Some generalities about basis number

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Remark

For any $k \in \mathbb{N}$, there exists a graph G with $\mathrm{bn}(G) = 3$, and a vertex $v \in V(G)$ such that $\mathrm{bn}(G - v) \geqslant k$.

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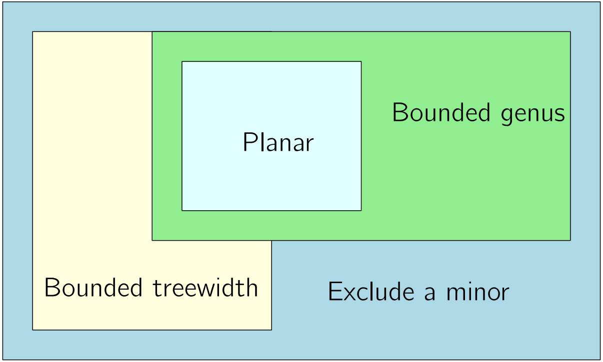
Minors



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Theorem (Geniet, G. 2026+)

There exists a function $f_{\min} : \mathbb{N} \to \mathbb{N}$ *such that for any graph* H, *any* H*-minor free graph* G *satisfies* $\mathrm{bn}(G) \leqslant f_{\min}(|H|)$.

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Our original proof gave $f_{\min}(t) = 2^{2^{O(t^2)}}$. Combining our proof with independant results of Miraftab, Morin and Yuditsky (2026+), we obtain $f_{\min}(t) = O(t^c)$ for some constant $c \leqslant 32210$.

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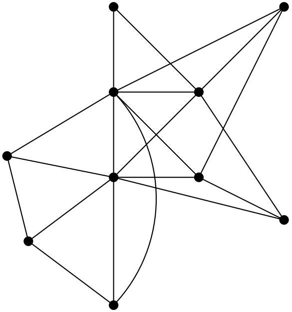
Corollary

Let $\mathcal{C}$ be a monotone class of graphs. Then $\mathcal{C}$ has bounded basis number if and only if all graphs in $\mathcal{C}$ exclude some fixed graph H as a minor.

Tree-decompositions



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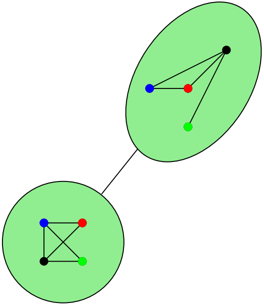

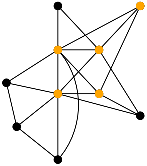

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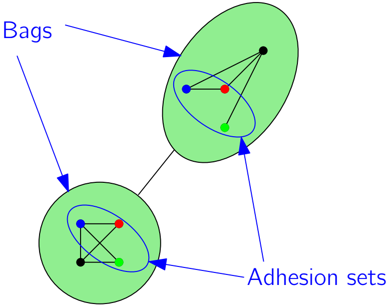

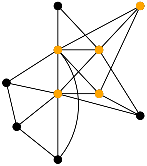

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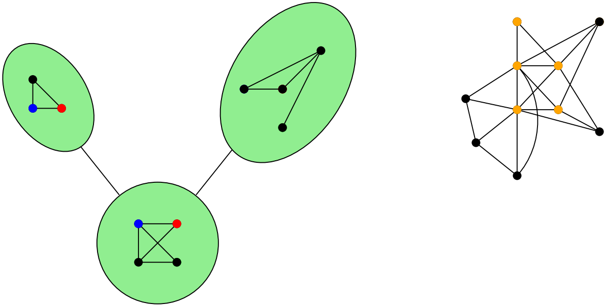

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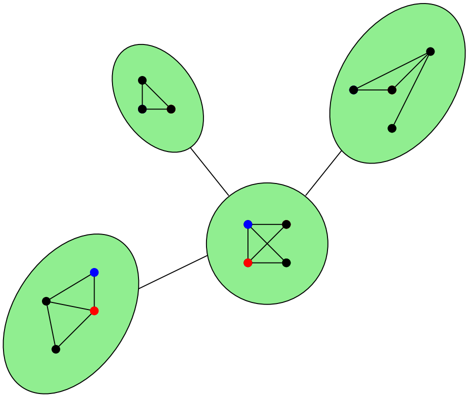

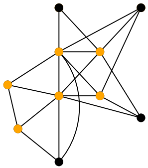

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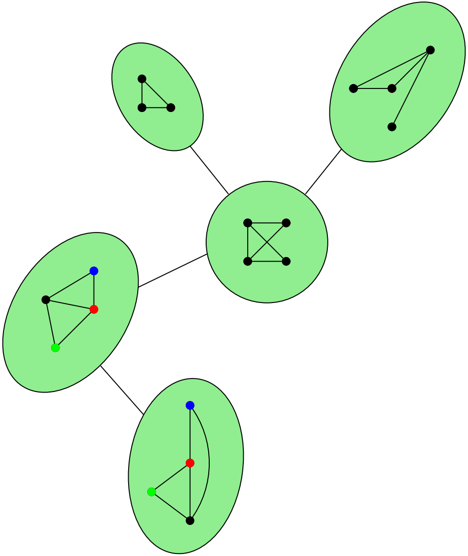

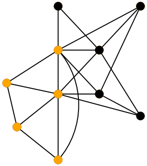

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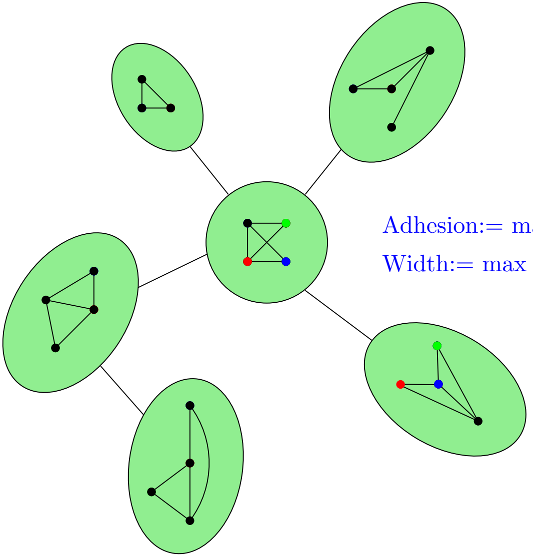
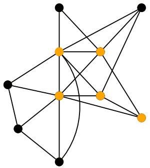
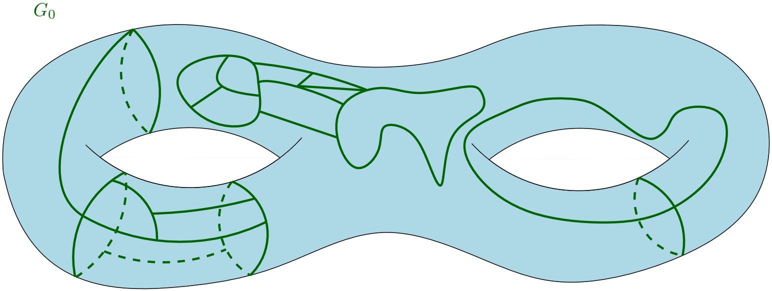




Adhesion:= max size of adhesion sets

Width:= max size bag - 1

Graph Minor Structure Theorem



(a, b)-almost embeddability

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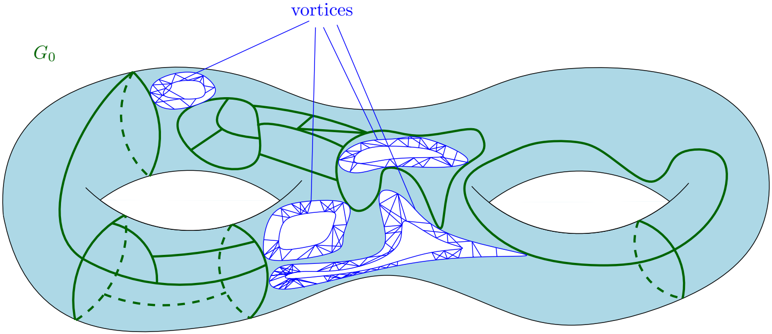

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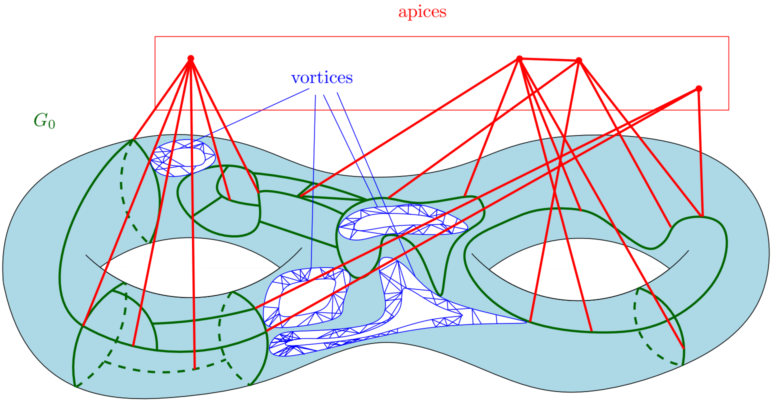

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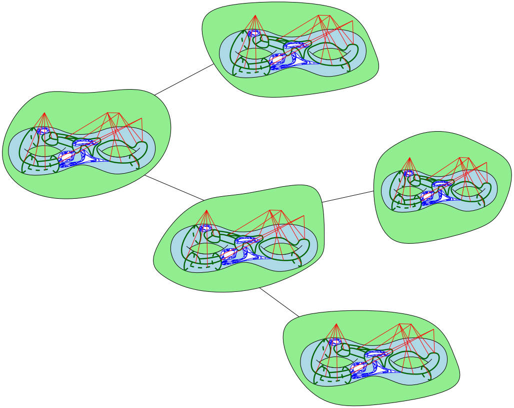
Theorem (Graph Minor Structure Theorem, Robertson, Seymour, 2003)

For every fixed H, there exists a, k, g every H-minor free graph G has a tree-decomposition of adhesion at most k, whose torsos are (a, k)-quasi embeddable in a surface of genus g.



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Gorsky, Seweryn and Wiederrecht (2025) proved that one can get $k, a \in O(|H|^{2300})$ and $g \in O(|H|^2)$.

Proof overview

Theorem (Geniet, G. 2026+)

There exists a function $f_{\min} : \mathbb{N} \to \mathbb{N}$ *such that for any graph* H*, any* H*-minor free graph* G *satisfies* $\mathrm{bn}(G) \leqslant f_{\min}(|H|)$*.*

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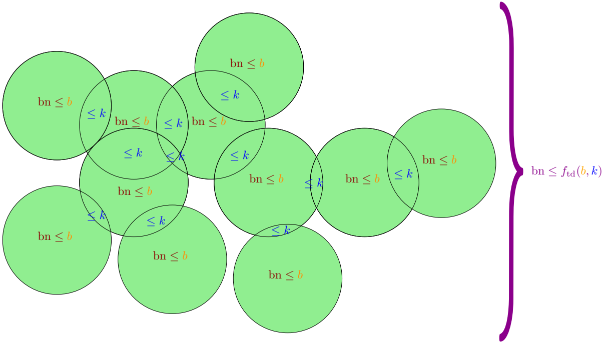
To prove our result using GMST, one must then know how to deal with:

- Tree-decompositions of bounded adhesion.
- Graphs quasi embeddable in a surface.

Theorem (Geniet, G. 2026+)

There exists $f_{\mathrm{td}} : \mathbb{N}^2 \to \mathbb{N}$ such that for each monotone graph class $\mathcal{G}$ with basis number at most b, every graph G with a tree-decomposition of adhesion at most k and whose torsos are all in $\mathcal{G}$ satisfies $\mathrm{bn}(G) \leqslant f_{\mathrm{td}}(b, k)$.



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Again, using [Miraftab, Morin, Yuditsky 2026+], $f_{\mathrm{td}}, f_{\mathrm{alm}}$ are polynomial.

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★ Step 4: Show the existence of f_{alm} and concludes using the GMST.

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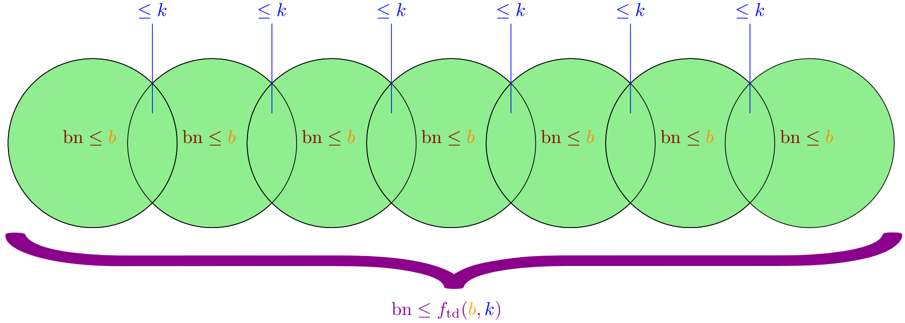

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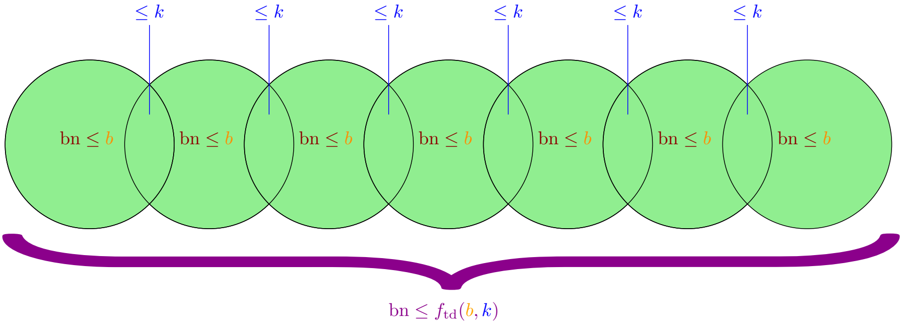

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Theorem (Miraftab, Morin, Yuditsky 2026+)

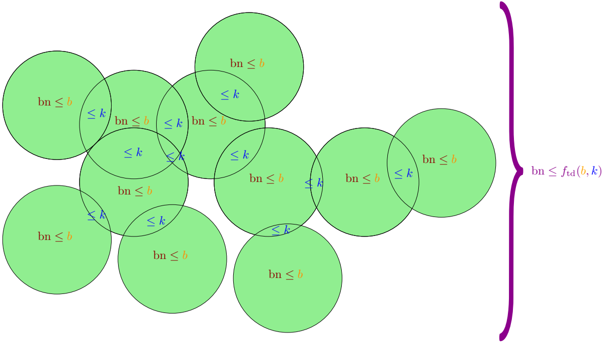
Any graph with pathwidth k has basis number at most $4k$.

Step 3

★ Step 3: Show the existence of f_{td} in the general case.

Theorem (Geniet, G. 2026+)

There exists $f_{\text{td}} : \mathbb{N}^2 \to \mathbb{N}$ such that for each monotone graph class $\mathcal{G}$ with basis number at most b, every graph G with a tree-decomposition of adhesion at most k and whose torsos are all in $\mathcal{G}$ satisfies $\text{bn}(G) \leqslant f_{\text{td}}(b, k)$.



Step 3: The bounded treewidth case

Lemma (Bojańczyk, Pilipczuk 2016 (simplified))

If G has a tree-decomposition (T, β) of width k, then there exists $X \subseteq V(T)$ such that the quotient $(T/X, \beta_X)$ tree-decomposition satisfies:

- *every bag admits a path-decomposition of width at most $3k+1$;*
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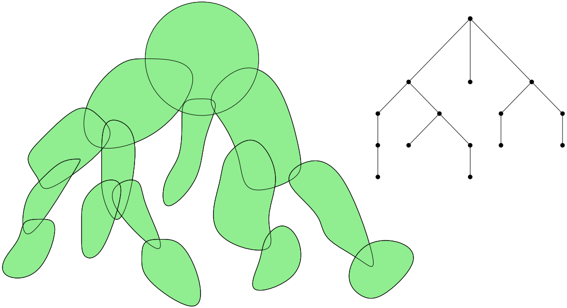

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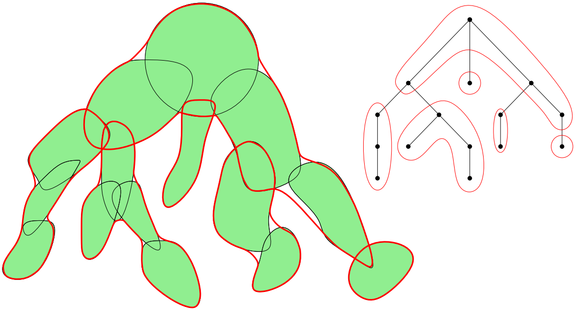

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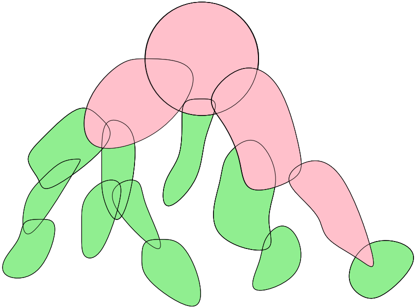

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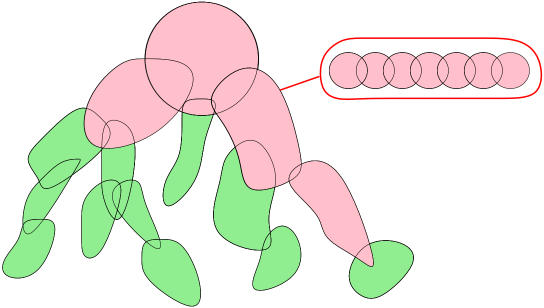

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Theorem (Us, using Miraftab, Morin, Yuditsky 2026+)

For every $k \geqslant 0$, every graph with treewidth k has basis number $O(k^5)$.

Lemma (Us, adapting Bojańczyk, Pilipczuk 2016 (simplified))

Let $\mathcal{G}$ be a monotone graph class. If G has a tree-decomposition (T, β) of adhesion k whose torsos are in $\mathcal{G}$, then there exists $X \subseteq V(T)$ such that the quotient $(T/X, \beta_X)$ tree-decomposition satisfies:

- *every bag admits a path-decomposition of adhesion at most $3k$, whose torsos are in $\mathcal{G}^{+2k}$;*
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Step 3: The general case

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For each monotone graph class $\mathcal{G}$ with basis number at most b, every graph G with a tree-decomposition of adhesion at most k and whose torsos are all in $\mathcal{G}$ satisfies $\mathrm{bn}(G) \leqslant (b + k\log^2 k)k^4$.

Step 4

★ Step 4: Show the existence of f_{alm} and concludes using the GMST.

Theorem (Geniet, G. 2026+)

There exists $f_{\text{alm}} : \mathbb{N}^3 \to \mathbb{N}$ *such that every graph* G *which is* (a, k)*-almost-embeddable in a surface of genus* g *satisfies* $\text{bn}(G) \leqslant f_{\text{alm}}(a, k, g)$.

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Theorem (Eppstein (2000) + Mazoit (2012))

Let G be a graph embedded in a surface $\mathbb{S}$ of genus g. Then

$$\text{tw}(G) = O(g \cdot \text{diam}(G^*)).$$

Conclusion

Theorem (Geniet, G. 2026+)

For any graph H, any H-minor free graph G satisfies

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Danke schön!