Structural and geometrical properties of highly symmetric graphs

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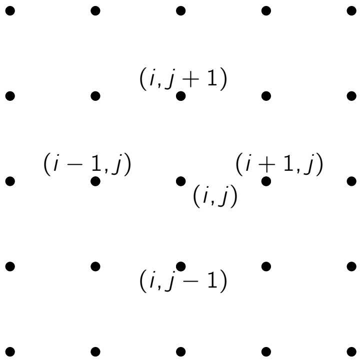
Université Grenoble Alpes, Laboratoire G-SCOP, France

PhD Thesis defense
9th July 2024

Cayley graphs

$(\Gamma, \cdot)$: group, S: finite set of generators. Cay(Γ, S) : graph with vertex set Γ and adjacencies $\{x, x \cdot a\}$ for every $x \in \Gamma, a \in S$.



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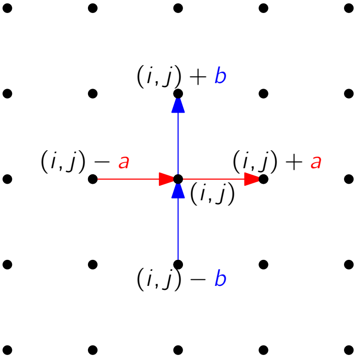

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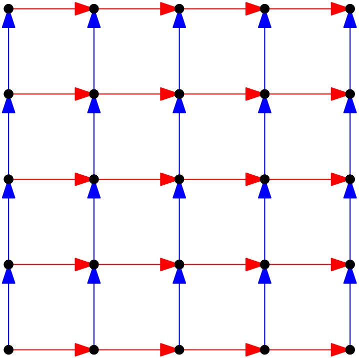

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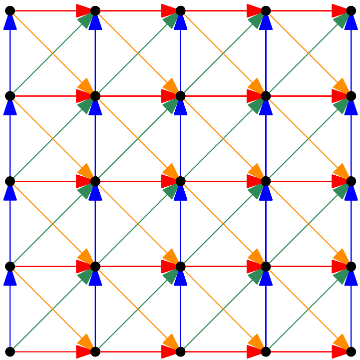

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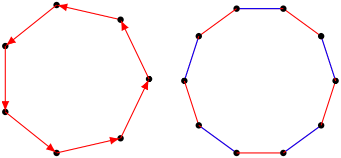

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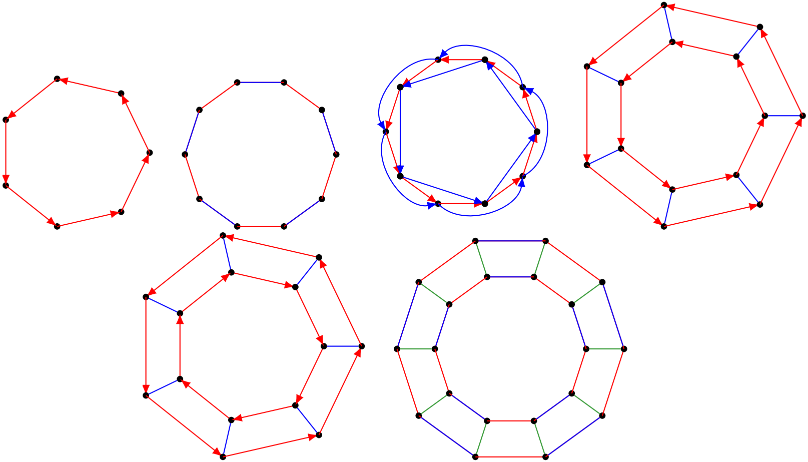

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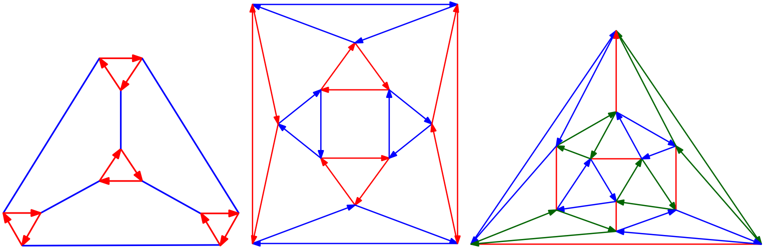

+ 15 others

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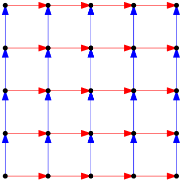

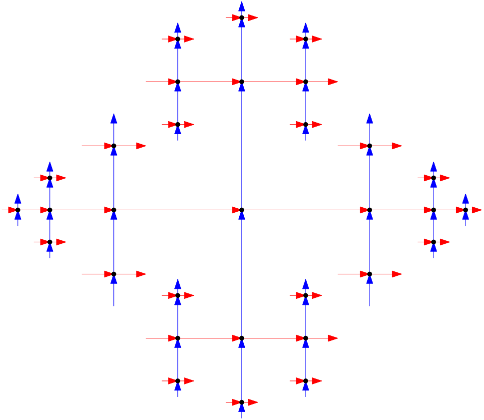

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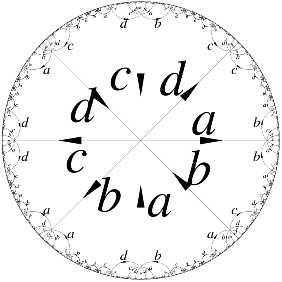

Image source: Yann Ollivier. A primer to geometric group theory.
`http://www.yann-ollivier.org/maths/primer.php`

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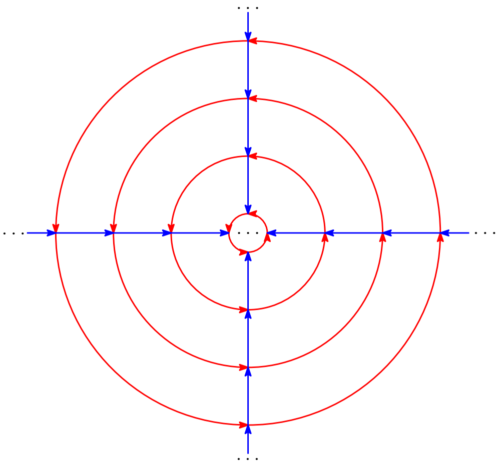
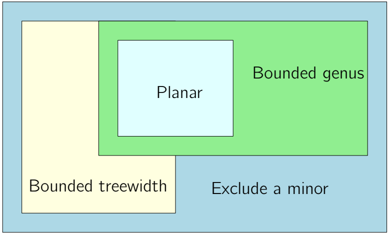
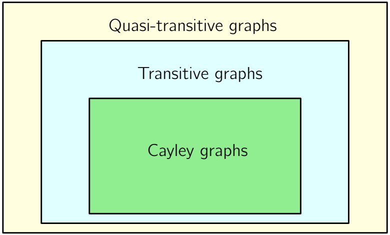


Plan of the presentation

In this presentation:

- Characterizations of classes of symmetric graphs defined by more general geometric properties.





- Connections with problems from symbolic dynamics.

Quasi-transitive graphs

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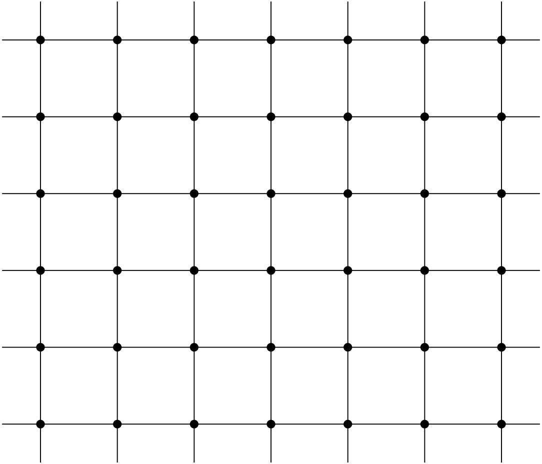

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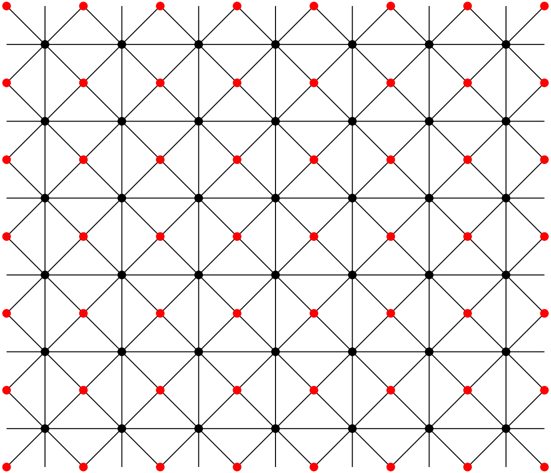

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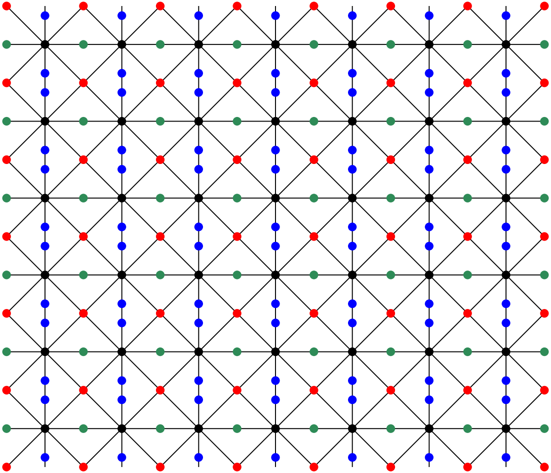

Minors

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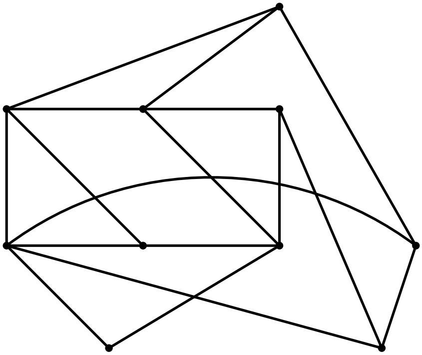

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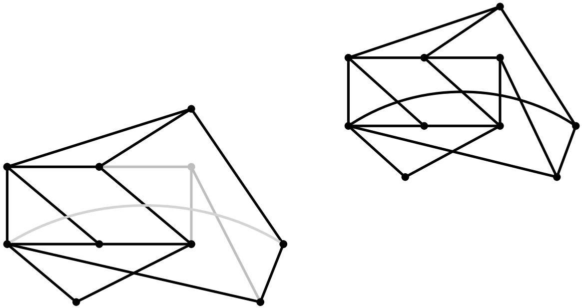

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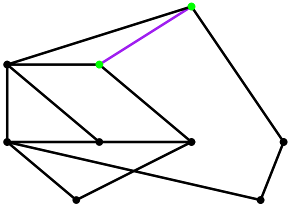

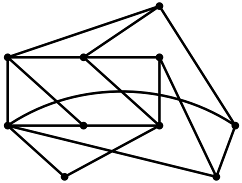

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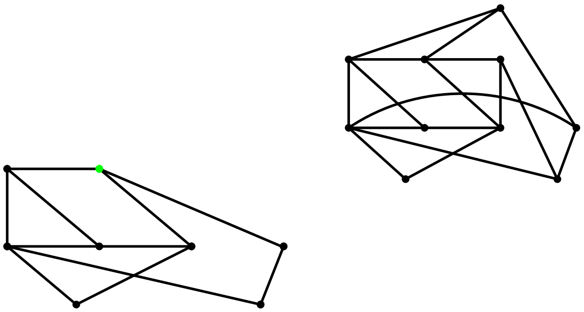

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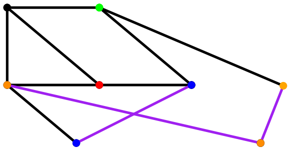

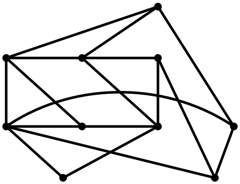

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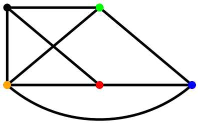
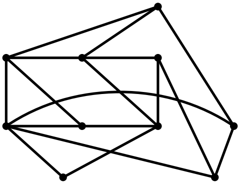
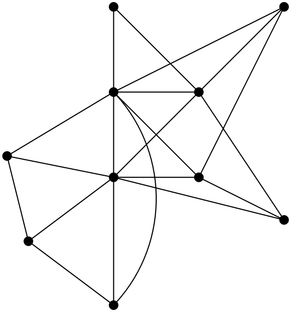




Tree-decompositions



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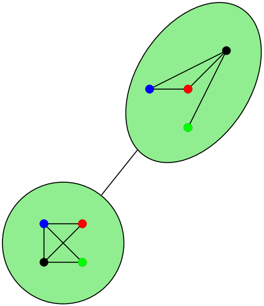

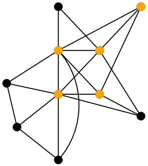

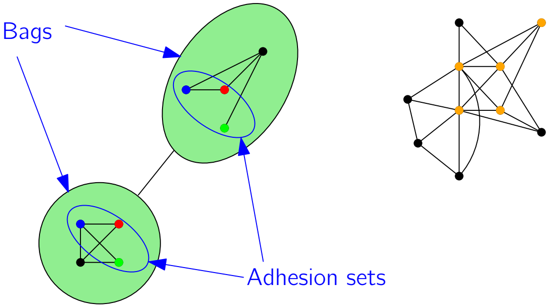
Bags
Adhesion sets

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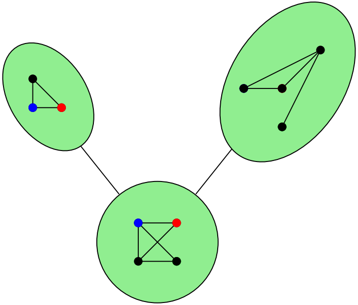

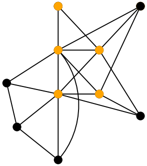

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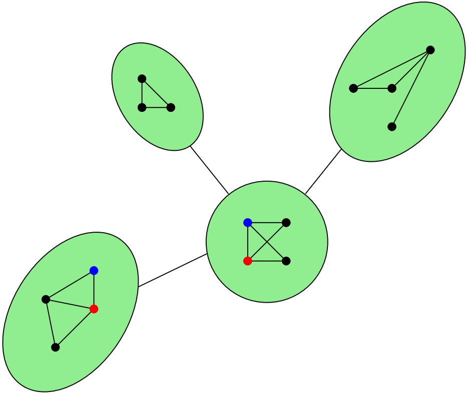

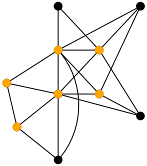

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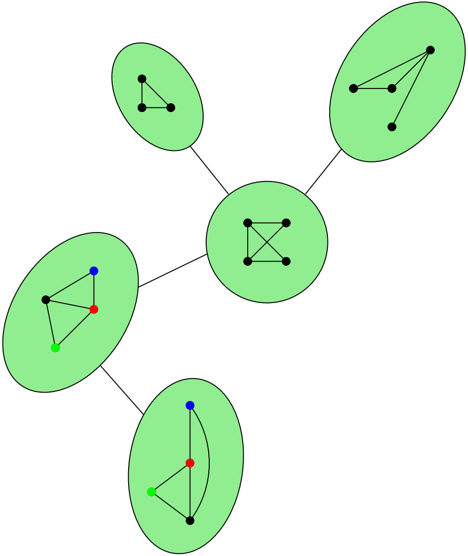

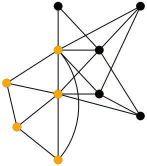

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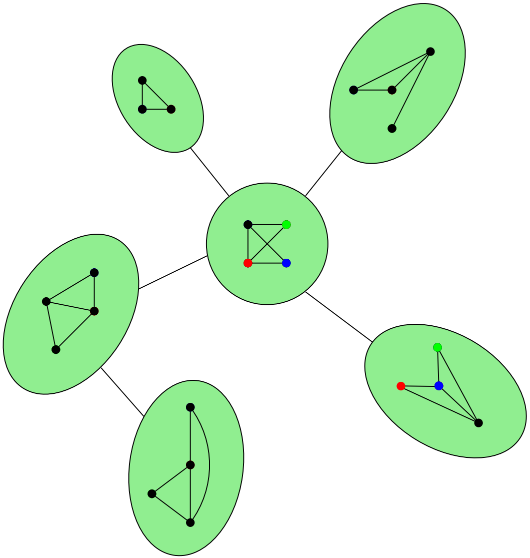

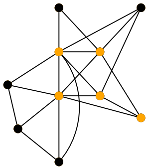

Graph Minor Structure Theorem

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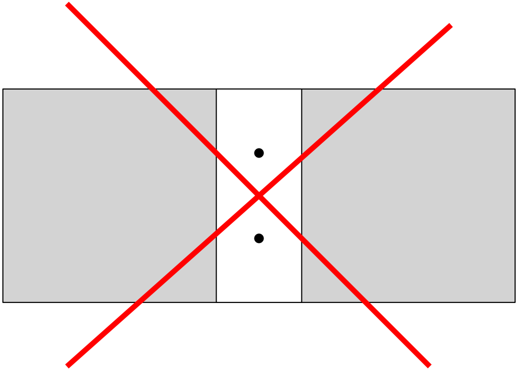
$\rightarrow$ None of these results are canonical.

Minors in quasi-4-connected graphs

G is quasi-4-connected if:

- G is 3-connected;
- the only separations of order 3 in G are between a single vertex and the remainder of the graph.



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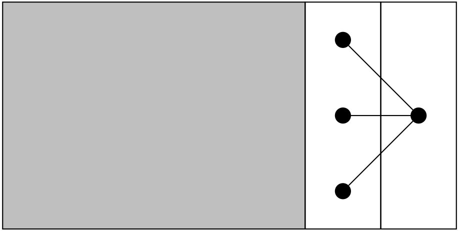

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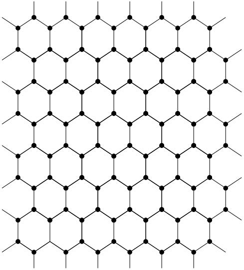
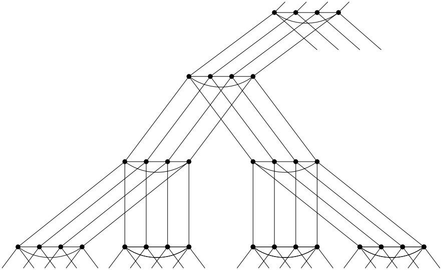
Theorem (Thomassen 1992)

If G is locally finite, quasi-transitive, quasi-4-connected and excludes K_∞ as a minor, then G is either planar or it has finite treewidth.





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Corollary (Thomassen 1992)

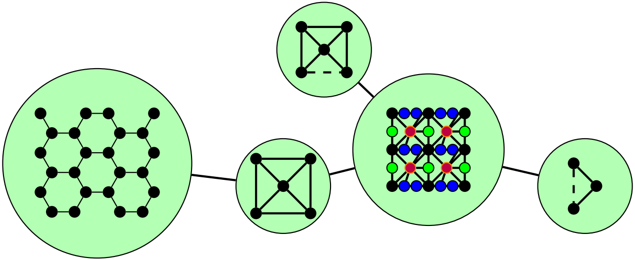
If G is locally finite, quasi-4-connected and quasi-transitive, and if G has every finite graph as a minor, then G has K_∞ as a minor.

→ Question (Thomassen 1992): Can we drop the quasi-4-connectivity condition?

Minor excluded quasi-transitive graphs

Theorem (Esperet, G., Legrand-Duchesne 2023 (finite/planar))

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



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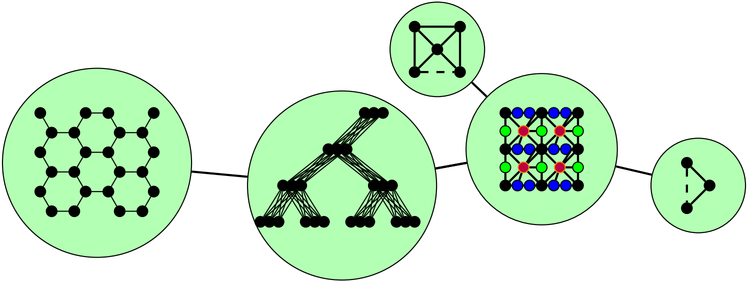
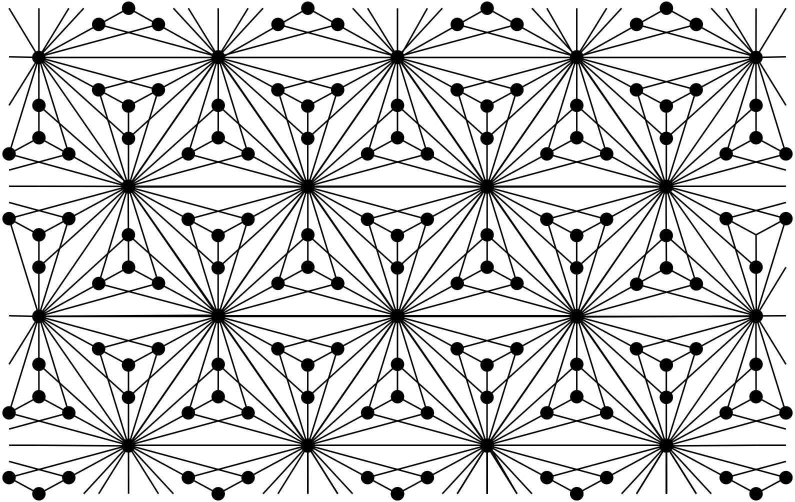


Simple example



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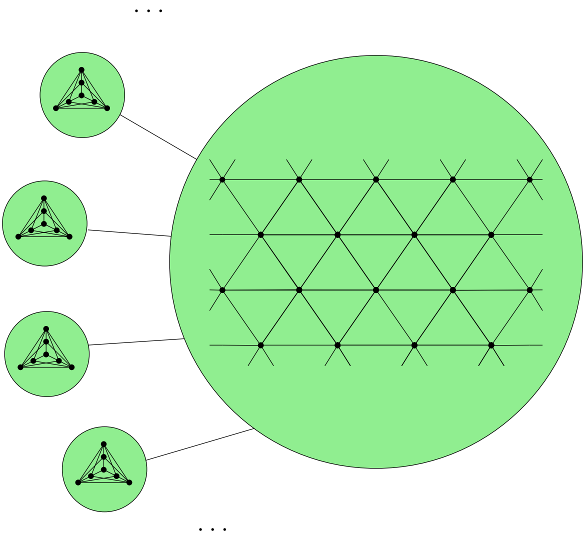

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Corollary

If G is locally finite, quasi-transitive and has every finite graph as a minor, then it also has K_∞ as a minor.

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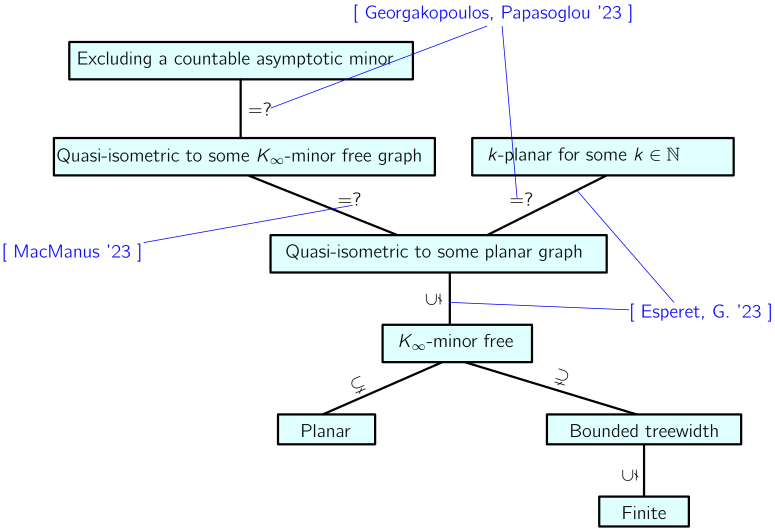
Proof based on results and methods from [Grohe '16] and [Carmesin, Hamann, Miraftab '22].

Beyond minor exclusion



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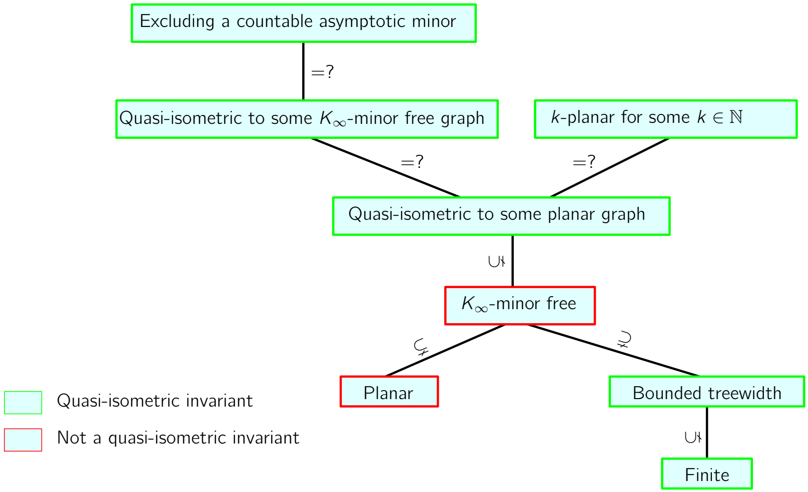
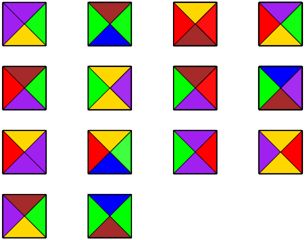


Wang tilings



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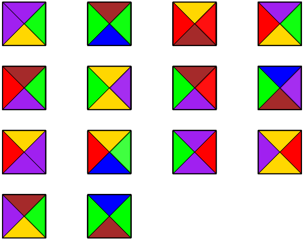

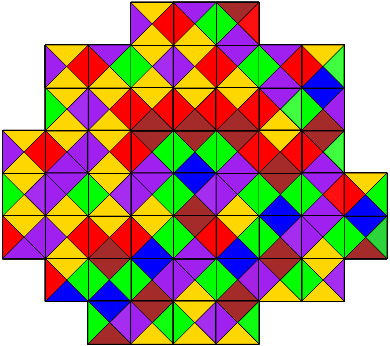

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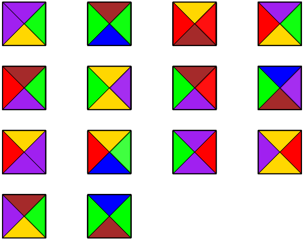
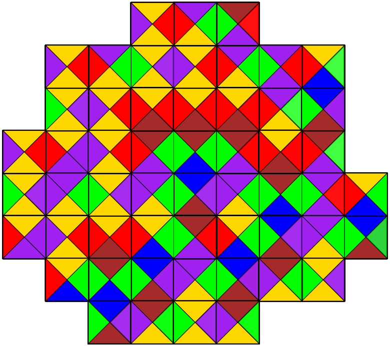


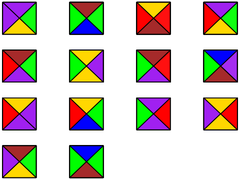


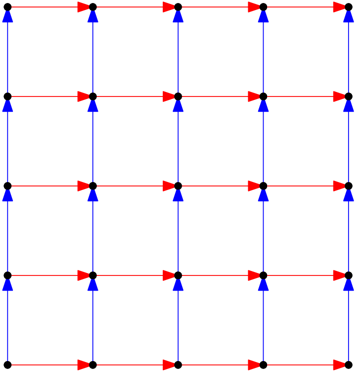
Two natural questions arise when given a finite set of Wang tiles:

- Does there exist a valid Wang tiling?
- If yes, does there exist a periodic one?

Wang tilings

Colors:





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$a := (1,0)$, $b := (0,1)$

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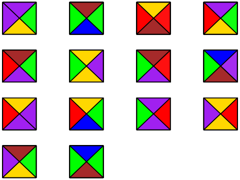
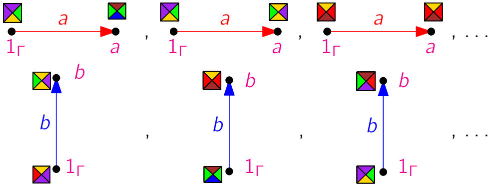
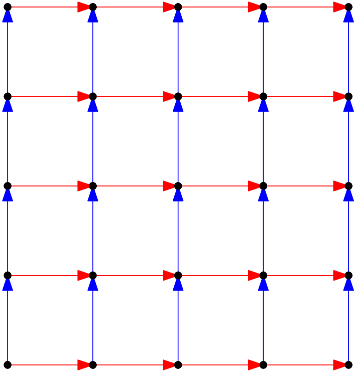


Local rules:





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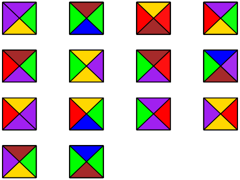

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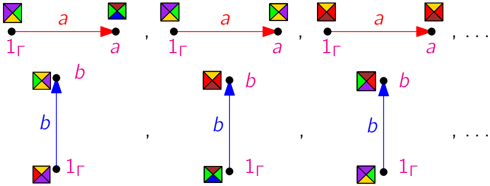

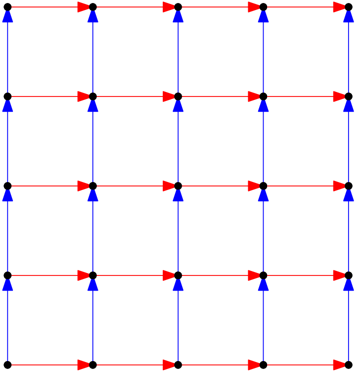

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→ Notions generalize to arbitrary Cayley graphs.

Domino problem on groups

Domino problem on $\mathrm{Cay}(\Gamma, S)$:

Input: a finite set of colors A and a finite set $\mathcal{R}$ of local rules.

Question: Is there a coloring $c : \mathrm{Cay}(\Gamma, S) \to A$ respecting $\mathcal{R}$?

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Intuition for graph theorists: bidimensionality → for every Cayley graph G:

- either G has bounded treewidth,
- or G has the infinite square grid as a minor.

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Similar conjecture (Carroll, Penland 2015) aiming at characterizing Cayley graphs of bounded pathwidth.

The minor excluded case

Corollary (Esperet, G., Legrand Duchesne, 2023)

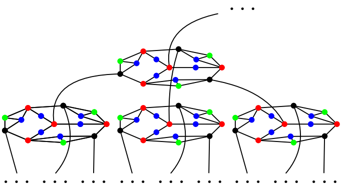
Both Ballier-Stein and Carroll-Penland conjectures are true for Cayley graphs excluding K_∞ as a minor.

Corollary (MacManus 2023)

Both Ballier-Stein and Carroll-Penland conjectures are true for Cayley graphs that are quasi-isometric to some planar graph.

Related questions (work in progress)

- Dynamics of SFTs corresponding to usual graph properties? e.g. proper colorings, matchings, orientations...



- Distinguishing weak and strong aperiodicity.

Thank you for your attention.