Coarse geometry of quasi-transitive graphs.

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Joint work with Louis Esperet

Jagiellonian University

10th Polish Combinatorial Conference.

Cayley graphs

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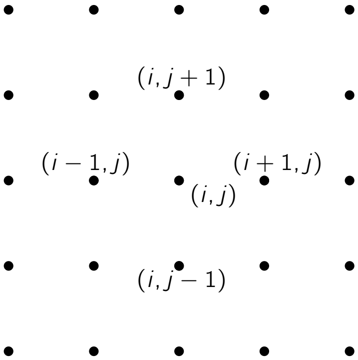

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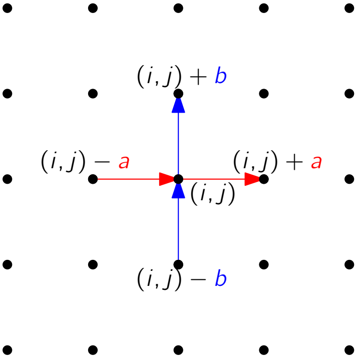

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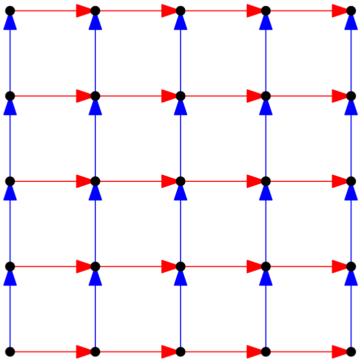

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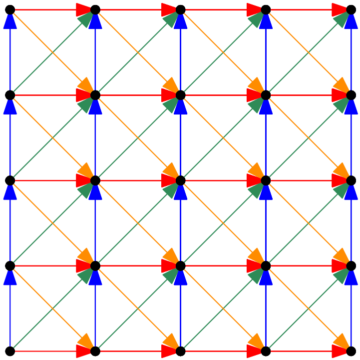

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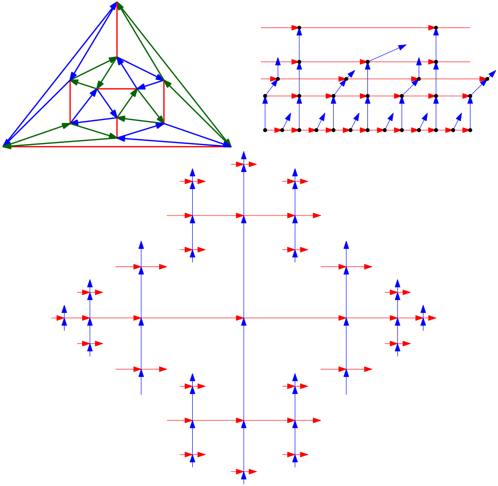

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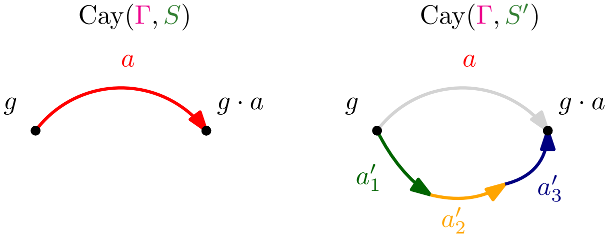

How are $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, S')$ related for two different finite generating sets S, S'?
Two (possibly infinite) graphs G, H are quasi-isometric to each other if there exist $f : V(G) \to V(H)$ and constants $A, B, C > 0$ such that:

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$$\forall x, y \in V(G), \frac{1}{A} d_G(x, y) - B \leqslant d_H(f(x), f(y)) \leqslant A d_G(x, y) + B,$$

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Geometric invariant := property preserved under taking quasi-isometries.

Remark

$\mathrm{Cay}(\Gamma, S)$ *and* $\mathrm{Cay}(\Gamma, S')$ *are quasi-isometric to each other.*

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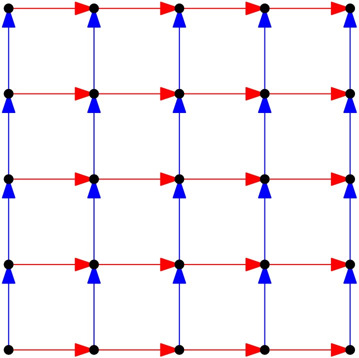

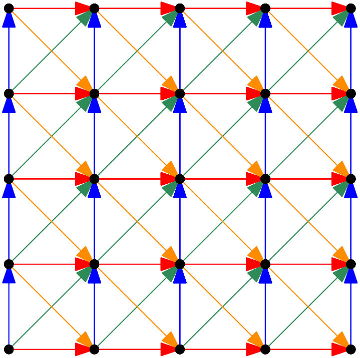

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Some geometric invariants:

- Finite diameter
- Number of ends
- Being hyperbolic
- Having finite treewidth (for graphs of bounded degree).

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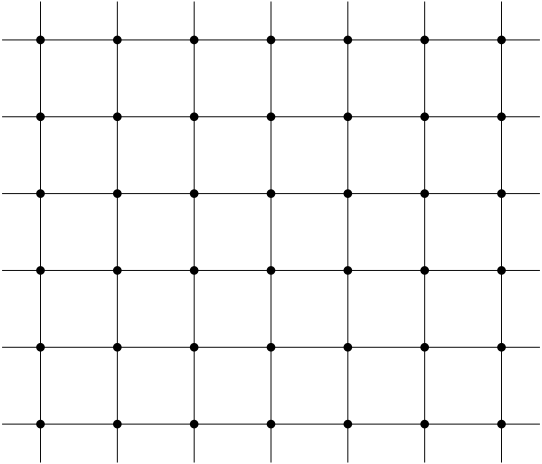

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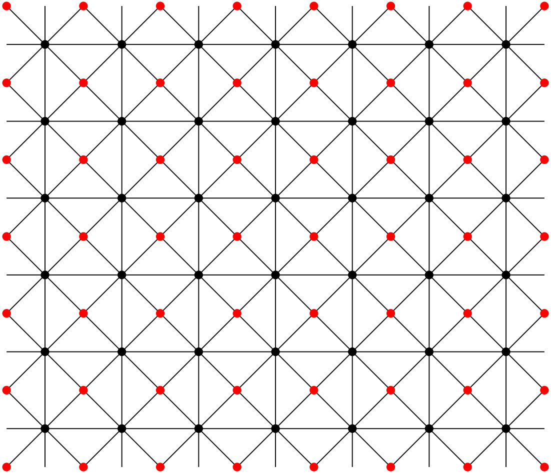

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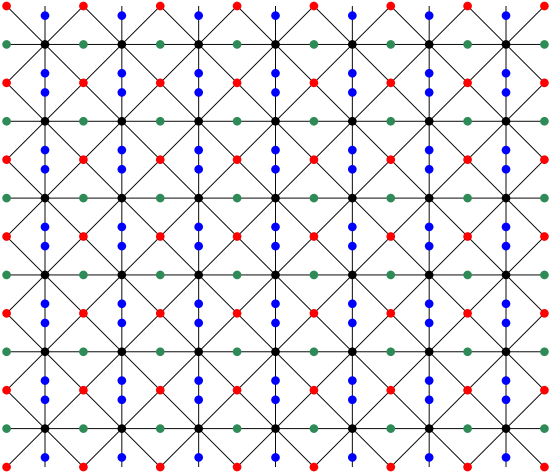

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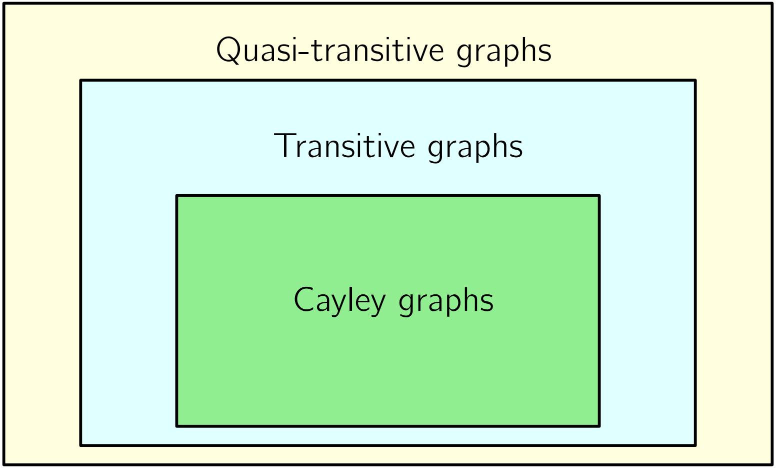

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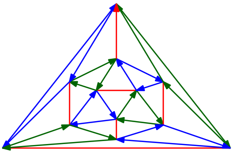
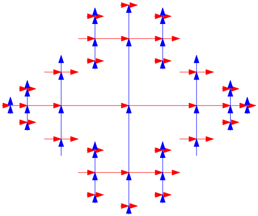
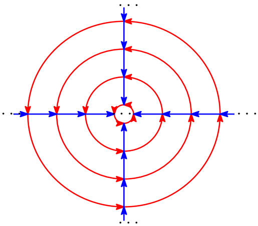
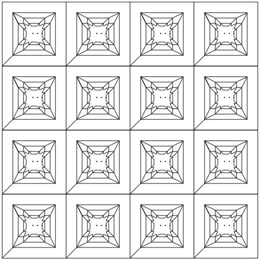

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- [MacManus 2023] Description extended to bounded degree quasi-transitive graphs being quasi-isometric to some planar graph.

Minors and quasi-isometries to planar graphs

Theorem (Esperet, G., 2024)

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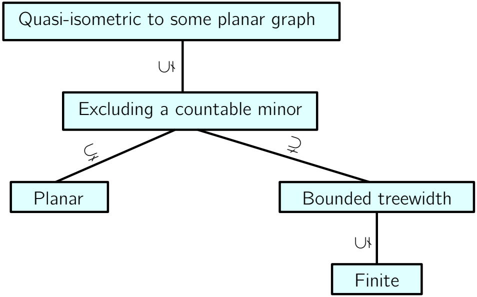

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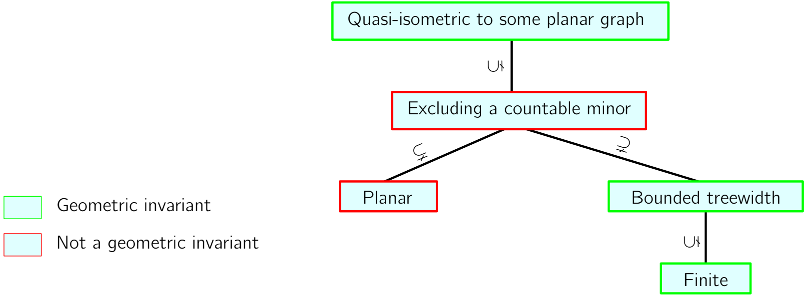

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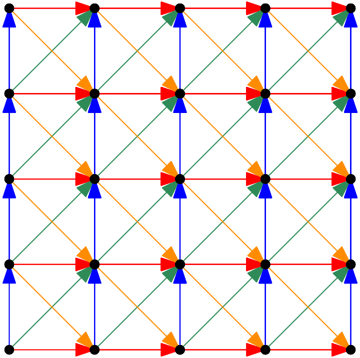

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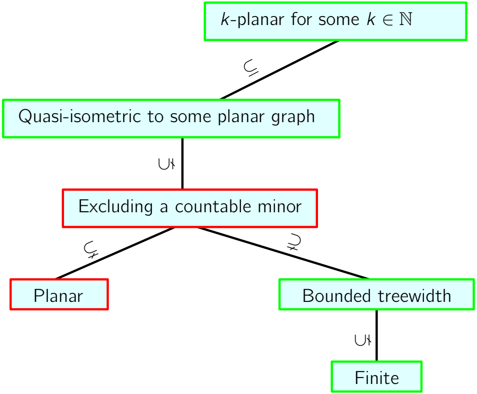

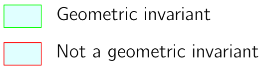

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Conjecture (Georgakopoulos, Papasoglou 2023)

Let G be a bounded degree quasi-transitive graph. Then G is quasi-isometric to a planar graph if and only if it is k-planar for some $k \in \mathbb{N}$.

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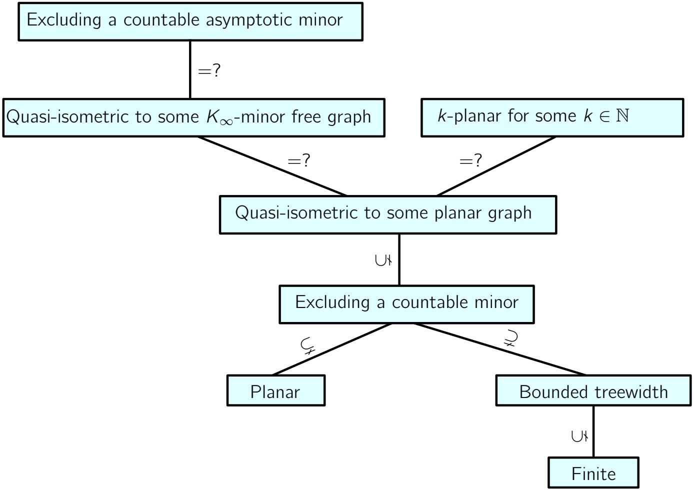

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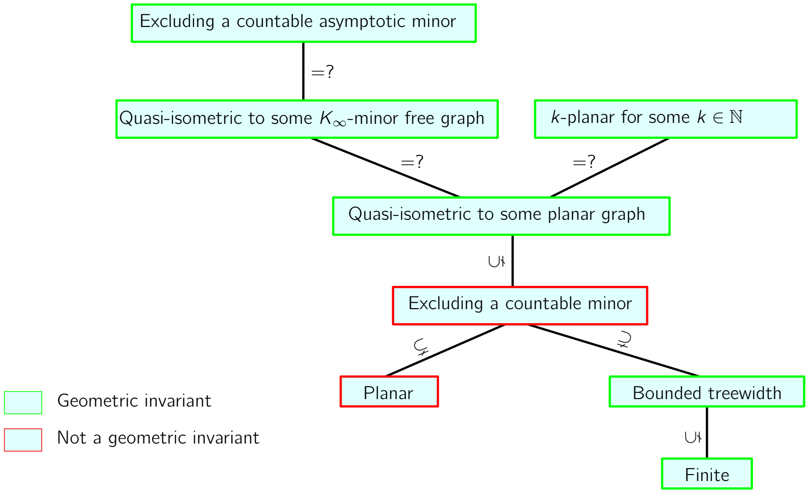

Thank you for your attention.