Twin-width IV: ordered graphs and matrices

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CoA 2022, Paris

Twin-width of unordered graphs

Definition (Contraction sequence)

Contraction sequence of $G = (V, E)$: sequence of *trigraphs* $(G = G_n, G_{n-1}, \dots, G_1)$ where G_{i-1} is obtained by identifying two vertices of G_i.

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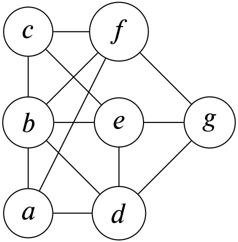
For every $X, Y \in V(G_i)$ put:

- An edge $XY \in E(G_i)$ if $G[X, Y]$ is a biclique;
- A nonedge in G_i if $G[X, Y]$ has no edge;
- A *red edge* $XY \in R(G_i)$ otherwise.

Twin-width of unordered graphs



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

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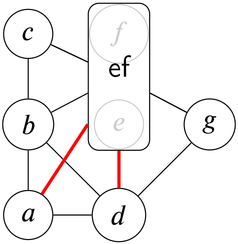

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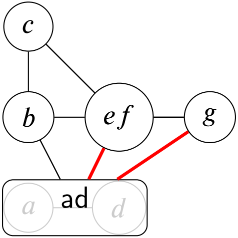

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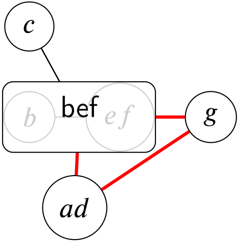

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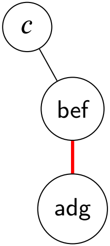

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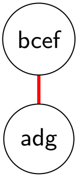

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Definition (Contraction sequence,twin-width)

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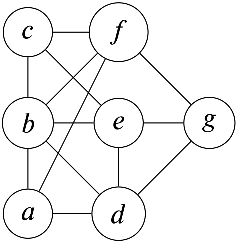
For every $X, Y \in V(G_i)$ put:

- An edge if $G[X, Y]$ is a biclique;
- A nonedge if $G[X, Y]$ has no edge;
- A *red edge* otherwise.

$(G_i)_i$ has *width* at most d if every G_i has red degree at most d.
The *twin-width* of G is the minimum width a contraction sequence of G could have.

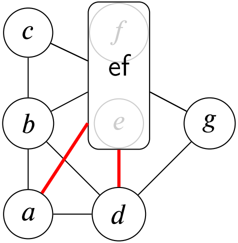
Twin-width of unordered matrices



	a	b	c	d	e	f	g
g	0	0	0	1	1	1	0
f	1	1	1	0	0	0	1
e	0	1	1	1	0	0	1
d	1	1	0	0	1	0	1
c	0	1	0	0	1	1	0
b	1	0	1	1	1	1	0
a	0	1	0	1	0	1	0

A graph together with one of its adjacency matrix.

Twin-width of unordered matrices



	a	b	c	d	e	f	g
g	0	0	0	1	1	1	0
f	1	1	1	0	0	0	1
e	0	1	1	1	0	0	1
d	1	1	0	0	1	0	1
c	0	1	0	0	1	1	0
b	1	0	1	1	1	1	0
a	0	1	0	1	0	1	0

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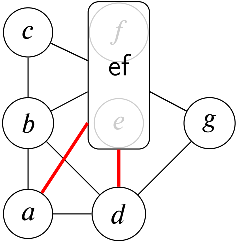

g	0	0	0	1	1	1	0
f	1	1	1	0	0	0	1
e	0	1	1	1	0	0	1
d	1	1	0	0	*r*	0	1
c	0	1	0	0	1	1	0
b	1	0	1	1	1	1	0
a	0	1	0	1	*r*	1	0
	a	b	c	d	e	f	g

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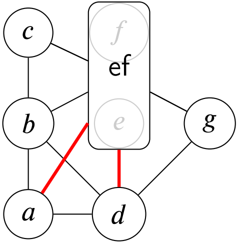

g	0	0	0	1	1	1	0
f	1	1	1	0	0	0	1
e	*r*	1	1	*r*	0	0	1
d	1	1	0	0	*r*	0	1
c	0	1	0	0	1	1	0
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a	0	1	0	1	*r*	1	0
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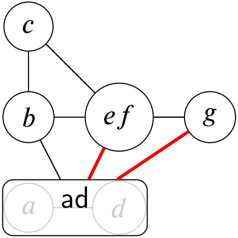


	a	b	c	d	e	f	g
g	r	0	0	1	1	1	0
f	1	1	1	0	0	0	1
e	r	1	1	r	0	0	1
d	r	1	0	0	r	0	1
c	0	1	0	0	1	1	0
b	1	0	1	1	1	1	0
a	r	1	0	1	r	1	0

Twin-width of unordered matrices

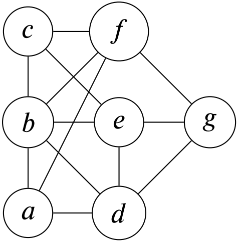
Twin-width naturally extends for matrices on finite alphabets.

Width of a sequence $\leftrightarrow$ maximum number of red entries on a row/column

Twin-width of ordered structures

Graphs are given together with a total order on their vertices.
Rows and columns indices of ordered matrices are totally ordered.

Twin-width of ordered structures



	a	b	c	d	e	f	g
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f	1	1	1	0	0	0	1
e	0	1	1	1	0	0	1
d	1	1	0	0	1	0	1
c	0	1	0	0	1	1	0
b	1	0	1	1	1	1	0
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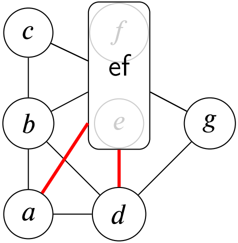
Left: Total order on $V(G)$: $a < b < c < d < e < f < g$. Right: the associated ordered adjacency matrix.

Twin-width of ordered structures



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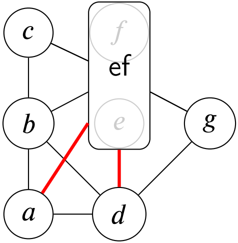

g	0	0	0	1	1	1	0
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d	1	1	0	0	*r*	0	1
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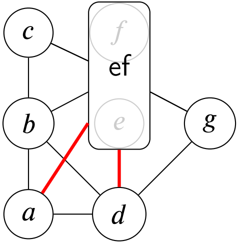

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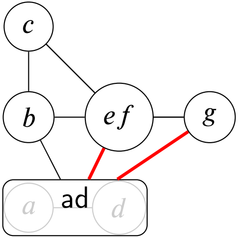

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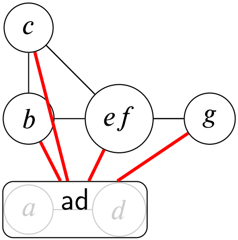

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Twin-width of ordered structures

Remark

A graph G has twin-width at most d if and only if there is a total ordering $<$ of $V(G)$ such that $(G,<)$ has twin-width at most d.

FO model checking on graphs

$\varphi \in \mathrm{FO}(E^{(2)})$: first order formula describing a graph problem.

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Example

$$\varphi = \exists x_1, \exists x_2, \ldots, \exists x_k, \forall x, \left(\bigvee_{i=1}^{k} x = x_i\right) \vee \left(\bigvee_{i=1}^{k} E(x, x_i)\right)$$

corresponds to k-Dominating Set problem.

FO model checking on graphs

$\varphi \in \mathrm{FO}(E^{(2)})$: first order formula describing a graph problem.

Definition

A class of graphs $\mathcal{C}$ is *FO-FPT* if there is an algorithm deciding for every $G \in \mathcal{C}$ whether $G \models \varphi$ in time $\mathcal{O}(f(|\varphi|) \cdot n^{\mathcal{O}(1)})$ for some computable f.

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Theorem (Bonnet, Kim, Thomassé, Watrigant '20)

There exists an algorithm that, given a graph G, a witness that $tww(G) \leq d$ and a formula φ, decides whether $G \models \varphi$ in time $\mathcal{O}(f(d, |\varphi|) \cdot n)$.

Algorithmic aspect of twin-width

Question (Fundamental question of twin-width)

Can we approximate twin-width?

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→ True for classes of ordered graphs/matrices!

Algorithmic aspect of twin-width for ordered structures

Theorem

There is an algorithm that, given an ordered $n \times n$ matrix M and an integer d, returns in time $\mathcal{O}(2^{2^{\mathcal{O}(d^2 \log(d))}} n^3)$:

- *"No" if $tww(G) > d$;*
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Theorem

Every hereditary class $\mathcal{C}$ of ordered graphs is FO-FPT if and only if it has bounded twin-width (unless $\mathrm{FPT} = \mathrm{AW}[*]$).

Small classes

Definition

A hereditary class of graphs (resp. ordered graphs) is *small* if it contains at most $n!c^n$ (resp. c^n) labeled graphs (resp. graphs) on n vertices.

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However true for classes of ordered matrices/graphs!

Twin-width and counting

Theorem (Matrix)

$\mathcal{M}$:class of ordered matrices closed under taking submatrices. Then exactly one of the following holds:

- *$\mathcal{M}$ has bounded twin-width and contains at most $2^{\mathcal{O}(n)}$ $n \times n$ matrices.*
- *$\mathcal{M}$ has unbounded twin-width and contains at least $\sum_{k=0}^{n} \binom{n}{k}^2 k! \geq n!$ $n \times n$ matrices.*

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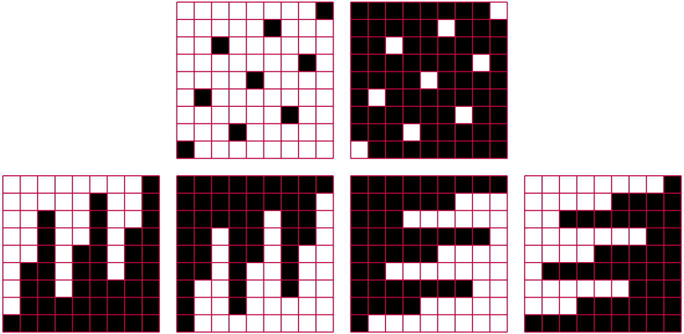
Theorem (Graph, conjectured in [Ballogh, Bollobás, Morris,'06])

$\mathcal{C}$:class of ordered graphs. Then exactly one of the following holds:

- *$\mathcal{C}$ has bouded twin-width and contains at most $2^{\mathcal{O}(n)}$ graphs of order n.*
- *$\mathcal{C}$ has unbounded twin-width and contains at least $\sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2k} k! \geq \lceil \frac{n}{2} \rceil !$ graphs of order n.*

Permutation matrices

6 different ways of encoding a single permutation.



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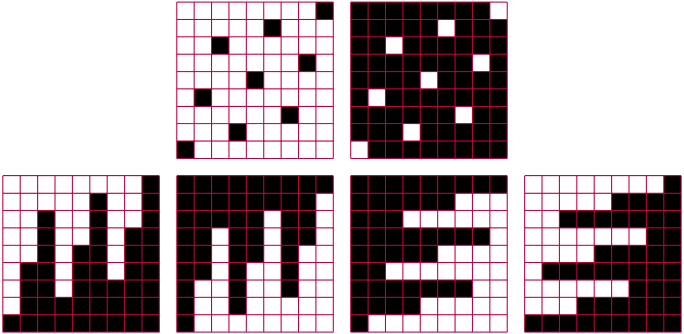
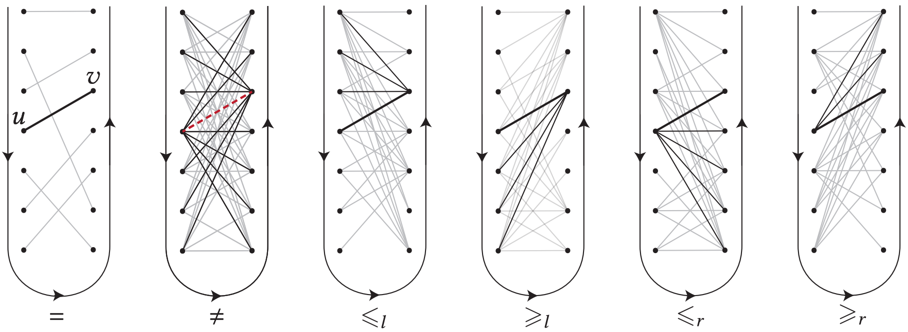


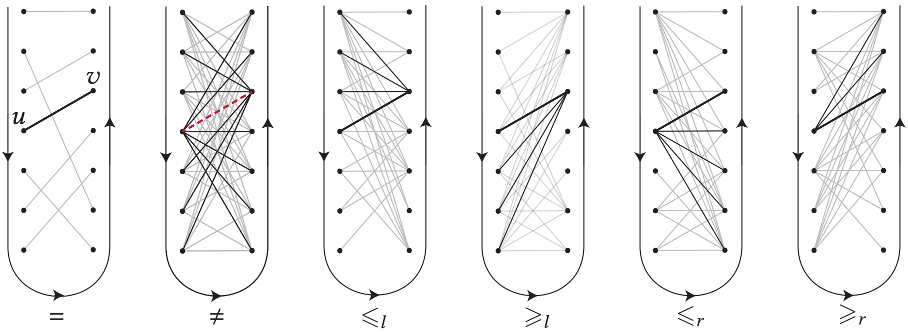
Theorem

A class of ordered matrices $\mathcal{M}$ has bounded twin-width if and only if it contains one of the six encodings of all the permutations.

Ordered matchings



Ordered matchings



Theorem

A class of ordered graphs $\mathcal{C}$ has bounded twin-width if and only if it contains one of the 24 encodings of all the ordered matchings or all the "ordered permutation graphs".

Interpretation: "Apply a first order formula φ on a graph G."

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Example

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Complement graph.

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Complement graph.

Example

$$\varphi(x, y) = \exists z, E(x, z) \land E(z, y)$$

Square graph.

Interpretations and transductions

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A graph class $\mathcal{C}$ is *dependant* (resp. monadically dependant) if the hereditary closure of every interpretation (resp. transduction) is not the class of all graphs.

→ Classes with bounded twin-width are monadically dependant.

Interpretations and transductions: the ordered case

Theorem

$\mathcal{C}$: class of ordered graphs. Then exactly one of the following holds:

- *$\mathcal{C}$ has bounded twin-width and is monadically dependant.*
- *$\mathcal{C}$ has unbounded twin-width and is not dependant.*

Main result

Theorem (Graph version)

Let $\mathscr{C}$ be a hereditary class of ordered graphs. The following are equivalent.

1. *$\mathscr{C}$ has bounded twin-width.*
2. *$\mathscr{C}$ is monadically dependent.*
3. *$\mathscr{C}$ is dependent.*
4. *$\mathscr{C}$ contains $2^{O(n)}$ ordered n-vertex graphs.*
5. *$\mathscr{C}$ contains less than $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$ ordered n-vertex graphs, for some n.*
6. *$\mathscr{C}$ includes neither one of 25 hereditary ordered graph classes $\mathscr{M}_{s,\lambda,\rho}$ nor all the ordered permutation graphs.*
7. *FO-model checking is fixed-parameter tractable on $\mathscr{C}$.*

Proof overview