Graphs with convex balls and groups acting on them

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Basic definitions

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d: Shortest path metric.
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- A set $X \subseteq V$ is *convex* if for every $u, v \in X$, $I(u, v) \subseteq X$.

Systolic graphs

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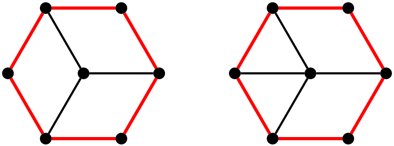
Definition

A subgraph H of G is *isometric* if for every $u, v \in V(H)$, $d_H(u, v) = d_G(u, v)$.



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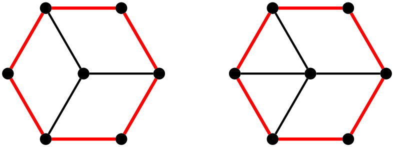

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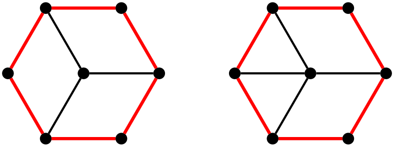

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Natural generalization of chordal graphs.

Combinatorial characterization of systolic graphs

Theorem (Chepoi, Soltan '83; Farber, Jamison '87)

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- G *is systolic;*
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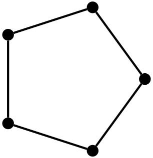
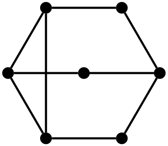
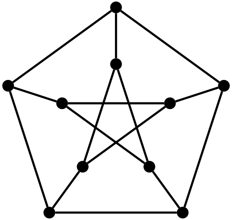
Definition

A *graph with convex balls* (or *CB-graph*) is a graph such that every $B_k(v)$ is convex for every $v \in V$, $k \geq 1$.

Some examples







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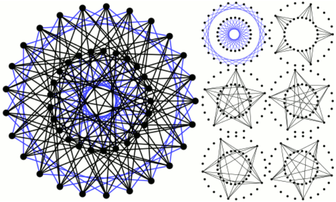

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From groups to graphs

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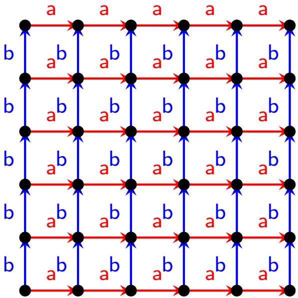
Definition

Cayley graph $\mathrm{Cay}(\Gamma, S)$ is the graph with vertex set Γ and adjacencies xy for every $x, y \in \Gamma$ such that $y \in S \cdot x$.

$\mathrm{Cay}(\mathbb{Z}^2, S)$, with $S = \{a, b\}$



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Definition

A group Γ acting geometrically by automorphisms on a CB-graph is called a *CB-group*.

Fellow traveler property

Definition

A path system $\mathcal{P}$ in G has the 2-*sided fellow traveler property* if there exists a constant $K > 0$ such that for every $\gamma, \gamma' \in \mathcal{P}$:

$$\forall i \geq 0, d(\gamma(i), \gamma'(i)) \leq K \max(d(\gamma(0), \gamma'(0)), d(\gamma(\infty), \gamma'(\infty))).$$

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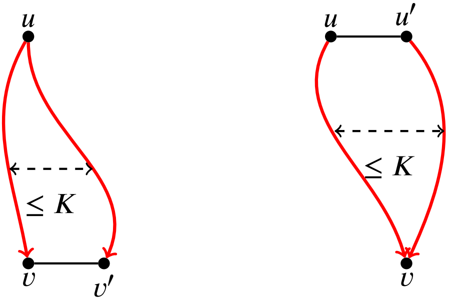

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$\Gamma =< S >$: finitely generated group. Assume $S = S^{-1}$.
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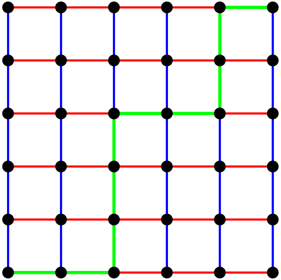

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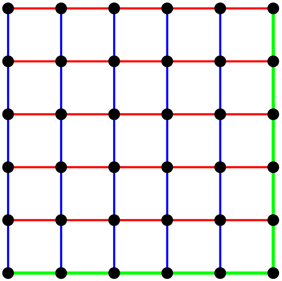

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$\Gamma =< S >$ is *biautomatic* if there exists a regular language $L \subseteq S^*$ that surjects onto Γ and which enjoys the 2-sided fellow traveler property in $\mathrm{Cay}(\Gamma, S)$.

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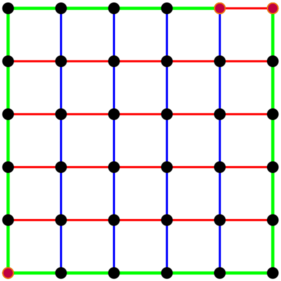

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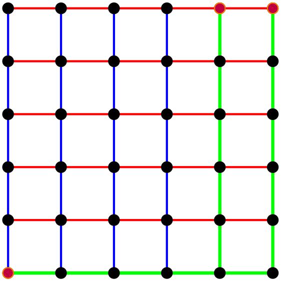

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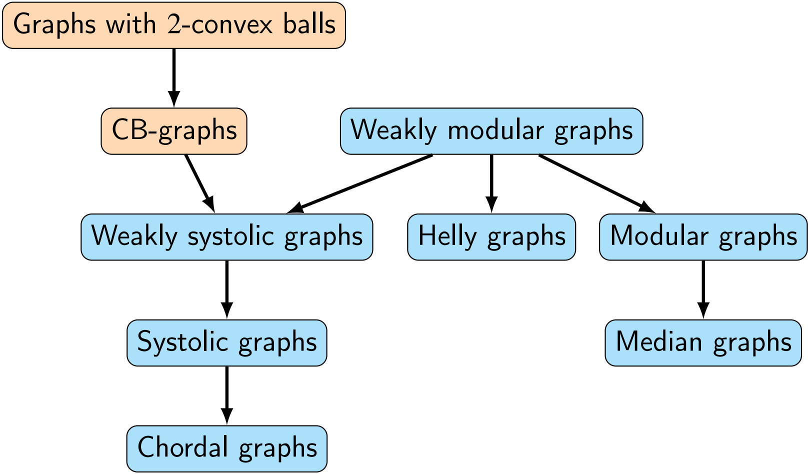

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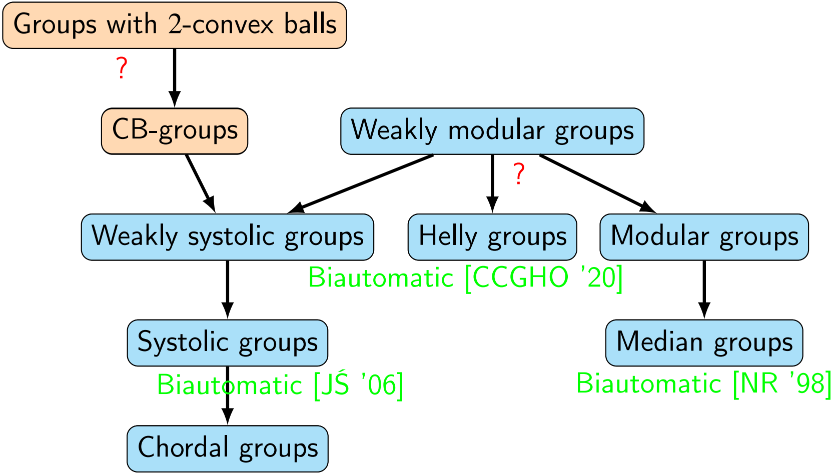

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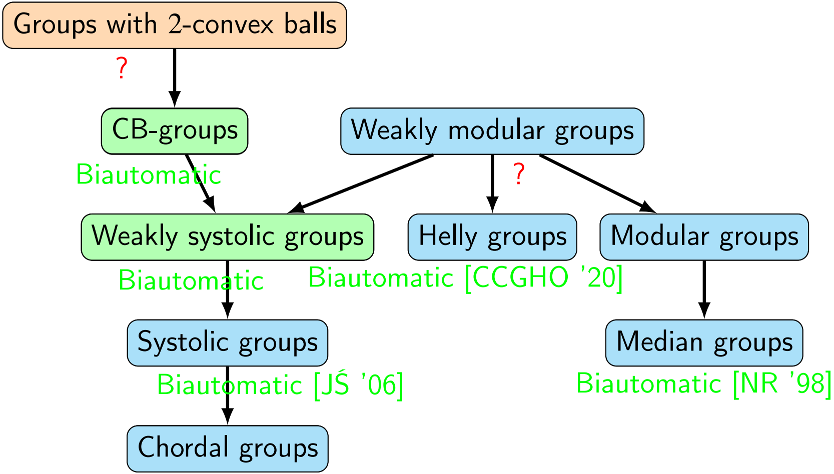

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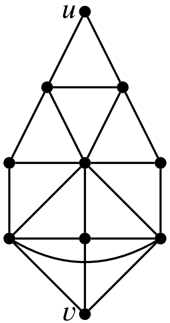
Theorem (Świątkowski, 06)

Let Γ be a group acting geometrically on a graph G, and $\mathcal{P}$ a path system in G such that:

1. *$\mathcal{P}$ is locally recognized;*
2. *there exists $v_0 \in V(G)$ such that any two vertices of the orbit $\Gamma \cdot v_0$ are connected by a path from $\mathcal{P}$;*
3. *$\mathcal{P}$ satisfies the 2-sided fellow traveler property.*

Then Γ is biautomatic.

Clique paths



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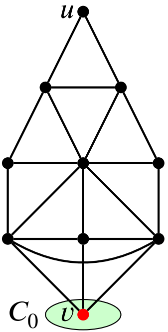

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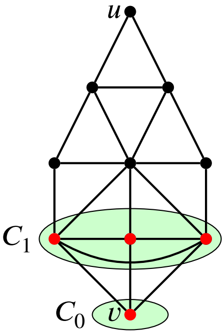

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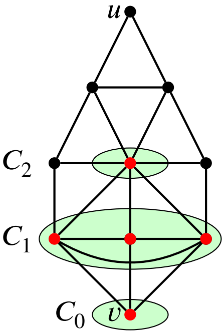

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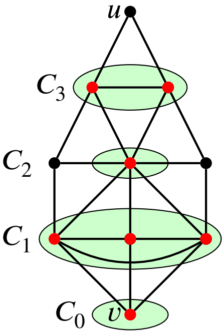

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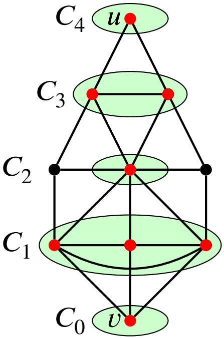

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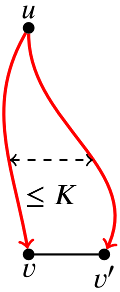

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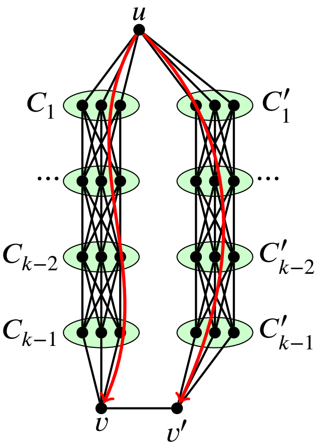

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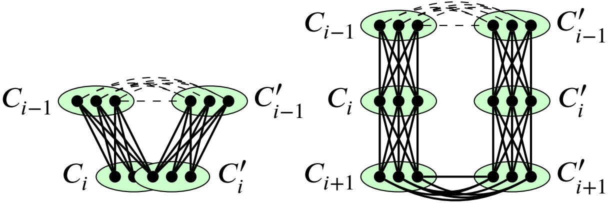

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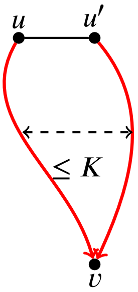

Other results about CB-graphs

Theorem (Local-to-global)

CB-groups are exactly graphs G such that:

- *The triangle-pentagon complex $X_{\triangle,\pentagon}(G)$ is simply connected, and*
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Theorem (metric triangles)

Every metric triangle of a CB-graph G is either equilateral, or can be completed into an induced C_5 of G.

Thank You

Question: Are CB-groups more general than weakly-systolic groups?